Inhomogeneous problems

Q. How do you kill a blue elephant?
A. With a blue elephant gun

Q. How do you kill a pink elephant?
A. Squeeze its trunk until it turns blue, and then shoot it with a blue elephant gun.

Q. How do you kill a white elephant?
A. Tickle it pink, then squeeze its trunk until it turns blue, and then shoot it with a blue elephant gun.

Q. How do you kill a yellow elephant?
A. Who's ever heard of a yellow elephant?

Up to now, we've dealt almost exclusively with problems for the wave and heat equations where the equations themselves and the boundary conditions are homogeneous. So a typical heat equation problem looks like

$$u_t = k\nabla^2 u \quad \text{for} \quad x \in D,\ t > 0$$

for a domain D (an interval on the line or region in the plane or in 3-space), subject to conditions like

$$u(\mathbf{x}, t) = 0 \text{ (or } \frac{\partial u}{\partial \mathbf{n}} = 0\text{) or some combination of these}$$

for $\mathbf{x}$ in the boundary of the region, and

$$u(\mathbf{x}, 0) = f(\mathbf{x}) \quad \text{in } D.$$

There are two new kinds of inhomogeneity we will introduce here. The first is an inhomogeneous boundary condition — so instead of being zero on the boundary, u (or $\partial u/\partial \mathbf{n}$) will be required to equal a given function on the boundary. The second kind is a "source" or "forcing" term in the equation itself (we usually say "source term" for the heat equation and "forcing term" with the wave equation), so we'd have

$$u_t = \nabla^2 u + Q(x, t)$$

for a given function Q.

The method we're going to use to solve inhomogeneous problems is captured in the elephant joke above. Up to now, we're good at "killing blue elephants" — that is, solving problems with inhomogeneous initial conditions. So if we encounter a "pink elephant" — a PDE with a source or forcing term — we'll do something to change the problem into a different problem without the forcing term but with inhomogeneous initial conditions (or different ones from the ones we had to begin with). And if we encounter a "white elephant" — a problem with inhomogeneous boundary values — we'll exchange the problem for one with a (different) forcing term (and different initial conditions), and then exchange that one for one whose only inhomogeneity is in the initial data.

That's the idea, and now we'll illustrate with examples.

Example 1 – a source term

We consider the following problem for the heat equation:

$$u_t = 5u_{xx} + e^{-t} \quad \text{for } 0 < x < 1 \text{ and } t > 0$$
$$u(0,t) = 0, \quad u(1,t) = 0$$
$$u(x,0) = x(1-x)$$

This problem has homogeneous boundary conditions ($u(0,t) = u(1,t) = 0$) but has the source term e^{-t} in the differential equation, so we deal with the source term first.

The way to deal with source (and forcing) terms is called the "method of eigenfunction expansions". In it, we take the non-t part of the differential equation (the u_{xx}) and consider the eigenfunctions of it taken together with the homogeneous boundary conditions. We're used to this, it's the problem for $X(x)$ that we've been encountering when we separate variables:

$$X'' + \lambda X = 0 \qquad X(0) = X(1) = 0.$$

We know that the eigenfunctions are $\sin n\pi x$ for $n = 1, 2, 3, \ldots$ and the corresponding values of λ are $n^2\pi^2$ for $n = 1, 2, 3, \ldots$.

To get rid of the source term, we seek a solution of the equation in the form

$$u(x,t) = \sum_{n=1}^{\infty} a_n(t) \sin n\pi x$$

So we're looking at the function $u(x,t)$ as a time-varying function on the interval $0 < x < 1$ — and so we're writing it as a Fourier sine series whose coefficients are allowed to vary with time. Because the boundary conditions are homogeneous, we are justified in plugging the series into the differential equation and moving the derivatives under the summation sign (my Math 360 conscience made me say that). So the differential equation becomes

$$\sum_{n=1}^{\infty} a_n'(t) \sin n\pi x = \sum_{n=1}^{\infty} -n^2\pi^2 a_n(t) \sin n\pi x + e^{-t}$$

We move the u_{xx} sum to the left side and combine the sums to obtain

$$\sum_{n=1}^{\infty} (a_n' + n^2\pi^2 a_n) \sin n\pi x = e^{-t}. \qquad (*)$$

Now, since we have a Fourier sine series (in x) on the left, we need to have a Fourier sine series (in x) on the right, so we expand the function e^{-t} as a Fourier sine series in x with coefficients that are functions of t:

$$e^{-t} = \sum_{n=1}^{\infty} b_n(t) \sin n\pi x$$

where

$$b_n = 2\int_0^1 e^{-t} \sin n\pi x \, dx = -\frac{2e^{-t}}{n\pi} \cos n\pi x \Big|_{x=0}^{x=1} = \frac{2e^{-t}}{n\pi}(1-(-1)^n) = \begin{cases} \dfrac{4e^{-t}}{n\pi} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

Now we can set the coefficients $a_n' + n^2\pi^2 a_n$ on the left side of equation (*) equal to the b_n's that we have found on the right, so we have the differential equations

$$a_n' + n^2\pi^2 a_n = 0 \quad \text{for even } n$$

and

$$a_n' + n^2\pi^2 a_n = \frac{4e^{-t}}{n\pi} \quad \text{for odd } n.$$

The general solution for even n is $a_n(t) = c_n e^{-n^2\pi^2 t}$ – this is also the "complementary solution" for odd n. But for the odd n we need a particular solution of

$$a_n' + n^2\pi^2 a_n = \frac{4e^{-t}}{n\pi},$$

which we find by the method of undetermined coefficients: Taking our cue from the right side of the equation, we guess that the particular solution is $a_n = Ae^{-t}$, and substitute this into the equation and get

$$(-1 + n^2\pi^2)Ce^{-t} = \frac{4e^{-t}}{n\pi},$$

so $C = 4/(n^3\pi^3 - n\pi)$ and so for odd n we have

$$a_n(t) = \frac{4e^{-t}}{n\pi(n^2\pi^2 - 1)} + c_n e^{-n^2\pi^2 t}.$$

OK, so far we have

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-n^2\pi^2 t} \sin n\pi x + \sum_{n \text{ odd}} \frac{4e^{-t} \sin n\pi x}{n\pi(n^2\pi^2 - 1)},$$

and for *any* (reasonable) choice of the c_n's this will satisfy the partial differential equation and the boundary conditions. So we're done with the pink elephant, and now we have to choose the c_n's so that the initial condition $u(x,0) = x(1-x)$ is satisfied.

From our expression for $u(x,t)$, we have that

$$u(x,0) = \sum_{n=1}^{\infty} c_n \sin n\pi x + \sum_{n \text{ odd}} \frac{4 \sin n\pi x}{n\pi(n^2\pi^2 - 1)},$$

and since we want this to be $x(1-x)$, we'll need the Fourier sine series for $x(1-x)$, which is

$$x(1-x) = \sum_{n \text{ odd}} \frac{8}{n^3\pi^3} \sin n\pi x$$

(we've done this one before). We conclude that for even n we have $c_n = 0$, and for odd n we have

$$c_n + \frac{4}{n\pi(n^2\pi^2 - 1)} = \frac{8}{n^3\pi^3},$$

so

$$c_n = \frac{4(n^2\pi^2 - 2)}{n^3\pi^3(n^2\pi^2 - 1)}.$$

Altogether, then, the solution of the problem is

$$u(x,t) = \sum_{n \text{ odd}} \frac{4(n^2\pi^2 - 2)}{n^3\pi^3(n^2\pi^2 - 1)} e^{-n^2\pi^2 t} \sin n\pi x + \sum_{n \text{ odd}} \frac{4e^{-t} \sin n\pi x}{n\pi(n^2\pi^2 - 1)},$$

which we could also write (substituting $2k+1$ for "n odd")

$$u(x,t) = \sum_{k=0}^{\infty} \left(\frac{4[(2k+1)^2\pi^2 - 2]}{(2k+1)^3\pi^3[(2k+1)^2\pi^2 - 1]} e^{-(2k+1)^2\pi^2 t} + \frac{4e^{-t}}{(2k+1)\pi[(2k+1)^2\pi^2 - 1]} \right) \sin(2k+1)\pi x.$$

This is the solution — the second term in the parentheses is devoted to the source term in the equation, and the first addresses the initial conditions.

Example 2 – the works

This time we'll consider a problem involving the wave equation:

$$u_{tt} = 4u_{xx} + (t+1)x \quad \text{for } 0 < x < \pi \text{ and } t > 0$$

$$u(0,t) = 0, \quad u(\pi,t) = \sin \alpha t$$

$$u(x,0) = 0, \quad u_t(x,0) = 0$$

This problem has both a forcing term $(t+1)x$ in the equation and an inhomogeneous boundary condition $u(\pi,t) = \sin \alpha t$ — we put the parameter α in the problem to illustrate something later. It won't matter that the initial conditions start out homogeneous, because in compensating for the other inhomogeneities we will introduce initial values and have to deal with them anyway.

When the boundary conditions are inhomogeneous, we always start with them, and try to find the simplest possible function $u_{\mathrm{bd}}(x,t)$ that satisfies them. In this case, since we have $u(0,t) = 0$ and $u(\pi,t) = \sin\alpha t$, it would make sense to use a function that is linear in x to match the boundary conditions for each t, that is,

$$u_{\mathrm{bd}}(x,t) = \frac{x}{\pi}\sin\alpha t.$$

Then we'll set

$$v(x,t) = u(x,t) - u_{\mathrm{bd}}(x,t).$$

We'll calculate what initial/boundary-value problem for v is implied by the one we're solving for u — it should and will have homogeneous boundary conditions. Then we'll solve the new problem for v and finally let $u = v + u_{\mathrm{bd}}$ to solve the original problem.

So we calculate: we have

$$v_{tt}(x,t) = u_{tt}(x,t) - (u_{\mathrm{bd}})_{tt}(x,t) = u_{tt}(x,t) + \frac{\alpha^2 x}{\pi}\sin\alpha t$$

and

$$v_{xx}(x,t) = u_{xx}(x,t) - (u_{\mathrm{bd}})_{xx}(x,t) = u_{xx}(x,t).$$

We can substitute this into the differential equation $u_{tt} = 4u_{xx} + (1+t)x$ to get

$$v_{tt} = 4v_{xx} + (1+t)x + \frac{\alpha^2 x}{\pi}\sin\alpha t.$$

The initial and boundary conditions for v will be those for $u - u_{\mathrm{bd}}$, so

$$v(0,t) = 0, \quad v(\pi,t) = 0$$

and, since $u_{\mathrm{bd}}(x,0) = 0$ and $(u_{\mathrm{bd}})_t(x,0) = \alpha x/\pi$, the initial conditions for v will be

$$v(x,0) = 0, \quad v_t(x,0) = -\frac{\alpha x}{\pi}.$$

Now (having dealt with the "white elephant") we're back in the situation of Example 1, with homogeneous boundary conditions and an inhomogeneous equation. We'll seek the function v in the form

$$v(x,t) = \sum_{n=1}^{\infty} a_n(t)\sin nx$$

because $\sin nx$ are the eigenfunctions of $X'' + \lambda X = 0$ with boundary conditions $X(0) = X(\pi) = 0$. We put this into the differential equation for v and obtain (after moving the $4v_{xx}$ term to the left side)

$$\sum_{n=1}^{\infty}(a_n'' + 4n^2 a_n)\sin nx = (1+t)x + \frac{\alpha^2 x}{\pi}\sin\alpha t = \left[(1+t) + \frac{\alpha}{\pi}\sin\alpha t\right]x.$$

So we need the Fourier sine series (in x, with coefficients that vary with t) for the function on the right side. It's not as bad as it looks, since the right side is just a multiple of x, and since

$$\frac{2}{\pi}\int_0^\pi x\sin nx\,dx = \frac{2}{\pi}\left(-\frac{x}{n}\cos nx\Big|_0^\pi + \int_0^\pi \frac{\cos nx}{n}\,dx\right) = (-1)^{n+1}\frac{2}{n},$$

the differential equation becomes

$$\sum_{n=1}^\infty (a_n'' + 4n^2 a_n)\sin nx = \sum_{n=1}^\infty \left((1+t) + \frac{\alpha^2}{\pi}\sin\alpha t\right)(-1)^{n+1}\frac{2}{n}\sin nx.$$

Setting the coefficients of $\sin nx$ equal to one another gives differential equations for the $a_n(t)$:

$$a_n'' + 4n^2 a_n = (-1)^{n+1}\frac{2}{n}\left((1+t) + \frac{\alpha^2}{\pi}\sin\alpha t\right).$$

As usual, we'll find a particular solution by the method of undetermined coefficients, and add to it the complementary solution $c_n\cos 2nt + d_n \sin 2nt$ of $a_n'' + 4n^2 a_n = 0$.

Here is the little complication introduced by the presence of the parameter α — and it has a physical meaning. If $\alpha \neq 2n$, then our guess for the particular solution will be

$$a_p = A + Bt + C\sin\alpha t + D\cos\alpha t$$

which gives us

$$\begin{aligned} a_p'' + 4n^2 a_p &= (-C\alpha^2\sin\alpha t - D\alpha^2\cos\alpha t) + 4n^2(A + Bt + C\sin\alpha t + D\cos\alpha t) \\ &= 4n^2(A+Bt) + (4n^2 - \alpha^2)(C\sin\alpha t + D\cos\alpha t) \end{aligned}$$

so to match the coefficients we must choose

$$A = B = \frac{(-1)^{n+1}}{2n^3} \quad C = (-1)^{n+1}\frac{2\alpha^2}{n\pi(4n^2-\alpha^2)} \quad \text{and } D = 0.$$

Therefore, if α is *not an even integer*, we can write our function v as:

$$v(x,t) = \sum_{n=1}^\infty (-1)^{n+1}\left(\frac{1+t}{2n^3} + \frac{2\alpha^2\sin\alpha t}{n\pi(4n^2-\alpha^2)} + c_n\cos 2nt + d_n\sin 2nt\right)\sin nx$$

and we're done with the pink elephant. All that's left is to match the initial conditions.

For $t = 0$, we have

$$v(x,0) = \sum_{n=1}^\infty (-1)^{n+1}\left(\frac{1}{2n^3} + c_n\right)\sin nx.$$

Then the initial condition $v(x,0) = 0$ tells us that we must choose $c_n = -1/(2n^3)$. The initial velocity is

$$v_t(x,0) = \sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{1}{2n^3} + \frac{2\alpha^3}{n\pi(4n^2 - \alpha^2)} + 2nd_n \right) \sin nx.$$

This is supposed to equal

$$v_t(x,0) = -\frac{\alpha x}{\pi} = \sum_{n=1}^{\infty} (-1)^n \frac{2\alpha}{\pi n} \sin nx$$

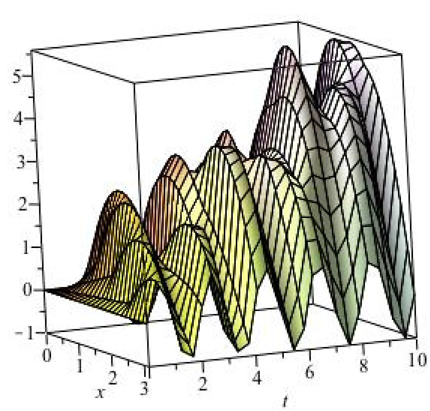
(from the Fourier series for x which we computed already). Therefore, equating corresponding coefficients gives us

$$\begin{aligned} d_n &= -\frac{1}{2n} \left(\frac{2\alpha}{\pi n} + \frac{1}{2n^3} + \frac{2\alpha^3}{n\pi(4n^2 - \alpha^2)} \right) \\ &= \frac{\alpha^3}{n^2\pi(\alpha^2 - 4n^2)} - \frac{\alpha}{\pi n^2} - \frac{1}{4n^4}. \end{aligned}$$

Now we can put these coefficients back into our expression for v, and add on u_{bd} to get the final solution of the problem (provided α is not an even integer):

$$\begin{aligned} u(x,t) = \frac{x}{\pi} \sin \alpha t + \sum_{n=1}^{\infty} (-1)^{n+1} \Bigg[& \frac{1+t}{2n^3} + \frac{2\alpha^2 \sin \alpha t}{n\pi(4n^2 - \alpha^2)} - \frac{1}{2n^3} \cos 2nt \\ & + \left(\frac{\alpha^3}{n^2\pi(\alpha^2 - 4n^2)} - \frac{\alpha}{\pi n^2} - \frac{1}{4n^4} \right) \sin 2nt \Bigg] \sin nx \end{aligned}$$

Here is a graph of the solution for $0 < x < \pi$ and $0 < t < 10$:



In the graph you can see the zero initial conditions and the sine wave on the boundary at $x = \pi$. This boundary condition causes the "waviness" of the solution. The forcing term causes the general arched shape of the solution to grow as t increases (from left to right).

We've so far ignored what happens if α is an even integer, but we will take that up now. For definiteness, let's put $\alpha = 12$. We don't have to reconstruct the entire solution — everything up until the middle of page 6, where we guessed the particular solution for the differential equation satisfied by $a_n(t)$ remains unchanged. And in fact, the rest of the solution, so all of the terms in the series for $u(x, t)$ *except the term for* $n = 6$ *(so* $2n = 12 = \alpha$*) remain unchanged* (except we can put $\alpha = 12$ in them). We need only to understand what changes for the function a_6.

The differential equation satisfied by a_6 is

$$a_6'' + 144a_6 = -\frac{1}{3}\left((1+t) + \frac{144}{\pi}\sin 12t\right).$$

The complementary solution is still $c_6 \cos 12t + d_6 \sin 12t$. But since the complementary solution has $\sin 12t$ in it, we cannot include this in our guess for the particular solution. In this case, our guess has to be

$$a_p = A + Bt + Ct\sin 12t + Dt\cos 12t$$

and we get

$$\begin{aligned} a_p'' + 144a_p &= [C(24\cos 12t - 144t\sin 12t) - D(24\sin 12t + 144t\cos 12t)] \\ &\quad + 144[A + Bt + Ct\sin 12t + Dt\cos 12t] \\ &= 144A + 144Bt + 24C\cos 12t - 24D\sin 12t \end{aligned}$$

so to match the coefficients we need

$$A = B = -\frac{1}{432} \quad C = 0 \quad \text{and } D = \frac{2}{\pi}.$$

Therefore, if $\alpha = 12$, we must write the $n = 6$ term of our function v as:

$$\left(-\frac{1+t}{432} + \frac{2t\cos 12t}{\pi} + c_6\cos 12t + d_6\sin 12t\right)\sin 6x$$

and what remains is to determine c_6 and d_6 from the initial conditions.

Setting $t = 0$ in this term yields the same equation as before, namely

$$-\frac{1}{432} + c_6 = 0$$

(well, we didn't put the complementary solution under the minus sign, but the result is the same: c_6 is what it was before). This derivative of this term with respect to t is

$$-\frac{1}{432} + \frac{2\cos 12t}{\pi} - \frac{24t\sin 12t}{\pi} - 12c_6\sin 12t + 12d_6\cos 12t.$$

So this term in the initial velocity is

$$-\frac{1}{432}+\frac{2}{\pi}+12d_6,$$

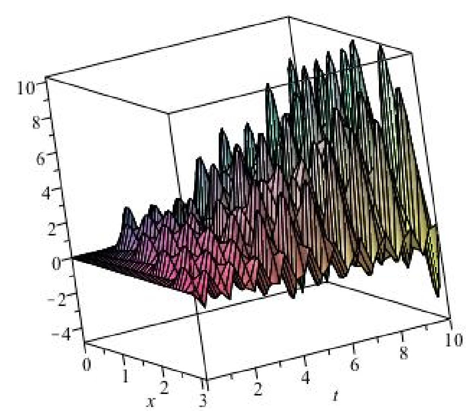
which is supposed to equal $4/\pi$ from Fourier series for the the initial condition $v_t(x,0)=-\alpha x/\pi$. Therefore

$$d_6=\frac{1}{5184}+\frac{1}{6\pi}.$$

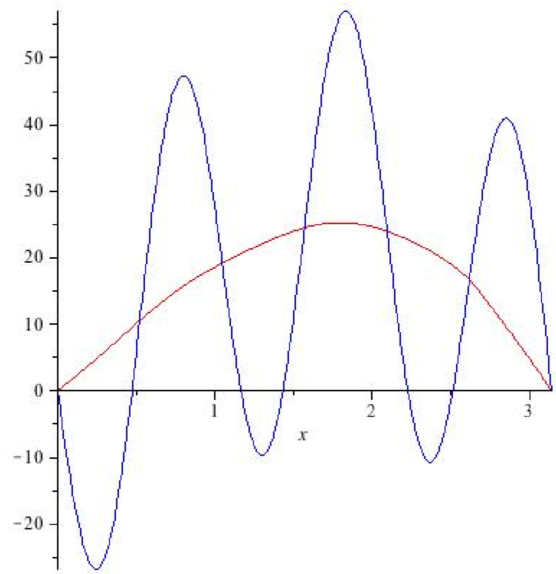
So the final solution of the problem for $\alpha=12$ is

$$\begin{aligned}u(x,t)=\frac{x}{\pi}\sin 12t+\sum_{\substack{n=1\\n\neq 6}}^{\infty}(-1)^{n+1}\Bigg[\frac{1+t}{2n^3}+\frac{288\sin 12t}{n\pi(4n^2-144)}-\frac{1}{2n^3}\cos 2nt\\+\left(\frac{1728}{n^2\pi(144-4n^2)}-\frac{12}{\pi n^2}-\frac{1}{4n^4}\right)\sin 2nt\Bigg]\sin nx\\+\left[-\frac{1+t}{432}+\frac{2t\cos 12t}{\pi}+\frac{\cos 12t}{432}+\left(\frac{1}{5184}+\frac{1}{6\pi}\right)\sin 12t\right]\sin 6x\end{aligned}$$

Note in the graph of the solution this time:



the scale on the u-axis goes from -5 to 10 as opposed to -1 to 5 like last time. This is because the effect of the wavy boundary condition is to build up a resonant wave along the string — because the boundary condition on the right side is oscillating with exactly the same frequency as one of the eigenfunctions (normal modes of vibration) of the string. This causes bigger and bigger vibrations of that frequency to build up in the string as time increases. It is this phenomenon that occurs, for example, when a singer shatters a glass by singing and sustaining a very high note. A better look at this would be provided by looking at the shape of the string for a fairly large positive value of t, say $t = 50$, for both the example with, say $\alpha = 4\pi$ and this one for $\alpha = 12$:



It's easy to tell which is which. An animation is more compelling and I'll try to get one up onto the webpage.

Example 3 – polar coordinates

We'll do one more example, this time in polar coordinates, to show how to carry out the procedure when Bessel functions are involved. This time, we'll solve the following problem for the heat equation on a disk of radius 5:

$$u_t = 3\nabla^2 u + 25 - x^2 - y^2 \quad \text{for } x^2 + y^2 < 25,\ t > 0$$
$$u(x, y, 0) = 0 \quad \text{for } x^2 + y^2 < 25$$
$$u = 20 \quad \text{on the boundary of the disk for all } t \geq 0.$$

From the form of the source term and the shape of the domain, it is clear that we should use polar coordinates — it's also the case that neither the source term, the initial data, nor the boundary data depend on θ, so the solution will also be independent of θ. Our problem is thus

$$u_t = \frac{3}{r}(ru_r)_r + 25 - r^2 \quad \text{fpr } r < 5,\ t > 0$$
$$u(r, 0) = 0, \quad \text{and} \quad u(5, 0) = 20.$$

As usual, we'll deal with the boundary condition first. But this time, because the problem involves the heat equation and the source term does not depend on t, there will be an equilibrium solution (a time-independent solution for which the heat being added by the source term is exactly balanced by flux out of the boundary). So rather than just subtracting off $u_{\text{bd}} = 20$, which would work, we'll find the equilibrium solution and subtract it from u instead.

The equilibrium solution satisfies the problem

$$\frac{3}{r}(ru_r)r + 25 - r^2 = 0, \quad u(0) \text{ finite}, \quad u(5) = 20.$$

We rewrite the differential equation as

$$3\left(u'' + \frac{1}{r}u'\right) = r^2 - 25,$$

which will become an inhomogeneous Cauchy-Euler equation if we multiply by r^2:

$$r^2u'' + ru' = \tfrac{1}{3}(r^4 - 25r^2).$$

For the complementary solution, we guess $u_c = r^\alpha$. Plug this in and discover that $\alpha = 0$ is a double root, so the general solution of the homogeneous equation $r^2u'' + ru' = 0$ is $u_c = c_1 + c_2 \ln r$. We can immediately set $c_2 = 0$ because we need $u(0)$ to be finite.

For the particular solution, we use the method of undetermined coefficients and guess $u_p = Ar^4 + Br^2$. Putting this into the equation gives

$$r^2u_p'' + ru_p' = r^2(12Ar^2 + 2B) + r(4Ar^3 + 2Br) = 16Ar^4 + 4Br^2.$$

For this to equal $\frac{1}{3}(r^4 - 25r^2)$ we need $A = \frac{1}{48}$ and $B = -\frac{25}{12}$. So far we have $u_{\text{eq}} = \frac{1}{48}r^4 - \frac{25}{12}r^2 + c_1$ and we need to choose c_1 to satisfy the condition $u(5) = 20$. So we solve

$$u_{\text{eq}}(5) = \frac{5^4}{48} - \frac{25 \cdot 5^2}{12} + c_1 = 20$$

and get $c_1 = \frac{945}{16}$.

So now we can set

$$v(r,t) = u(r,t) - u_{\text{eq}}(r) = u(r,t) - \left(\tfrac{1}{48}r^4 - \tfrac{25}{12}r^2 + \tfrac{945}{16}\right).$$

The bonus we will reap for doing this is to get rid of both the boundary values and the source term:

$$\begin{aligned} v_t - 3\nabla^2 v &= u_t - 3\nabla^2 u - 3\nabla^2\left(\tfrac{1}{48}r^4 - \tfrac{25}{12}r^2 + \tfrac{945}{16}\right) \\ &= u_t - 3\nabla^2 u - 25 + r^2 \\ &= 0 \end{aligned}$$

because $3\nabla^2 u_{\text{eq}} = r^2 - 25$ (you should check this!). The boundary values of v are zero (i.e., $v(5,t) = 0$) because the boundary values of u_{eq} are the same as those of u, namely $u_{\text{eq}}(5,t) = 20$. The only distressing thing is the initial values of v:

$$v(r,0) = u(r,0) - u_{\text{eq}}(r) = -\tfrac{1}{48}r^4 + \tfrac{25}{12}r^2 - \tfrac{945}{16}.$$

To recap, the problem we have to solve for v is

$$v_t = 3\left(v_{rr} + \frac{1}{r}v_r\right), \quad v(5,t) = 0, \quad v(r,0) = -\tfrac{1}{48}r^4 + \tfrac{25}{12}r^2 - \tfrac{945}{16}.$$

We want to use a series of r-eigenfunctions with coefficients that are functions of t the way we did for the previous examples. So we first need the eigenvalues and eigenfunctions of the problem

$$R'' + \frac{1}{r}R' + \lambda R = 0, \quad R(0) \text{ finite}, \quad R(5) = 0$$

We multiply the differential equation by r^2 and recognize

$$r^2R'' + rR' + \lambda r^2 R = 0$$

as Bessel's equation of order zero, which has general solution

$$R(r) = cJ_0(\sqrt{\lambda}\, r) + dY_0(\sqrt{\lambda}\, r).$$

We immediately set $d = 0$ because $Y_0(r)$ goes to $-\infty$ as r goes to 0. Since we also need $R(5) = 0$, we conclude that

$$R(r) = J_0\left(\frac{z_n r}{5}\right) \quad \text{and} \quad \lambda = \frac{z_n^2}{25}$$

where we write z_n for the nth positive zero of $J_0(x)$ (we write $z)n$ instead of z_{0n} since we won't be using any Bessel functions besides J_0 for this problem).

So we seek $v(r,t)$ in the form:

$$v(r,t) = \sum_{n=1}^{\infty} a_n(t) J_0\left(\frac{z_n r}{5}\right).$$

We have that

$$v_t(r,t) = \sum_{n=1}^{\infty} a_n'(t) J_0\left(\frac{z_n r}{5}\right)$$

and, because $J_0\left(\frac{z_n r}{5}\right)$ is a solution of the equation $R'' + \frac{1}{r}R' = -\frac{z_n^2}{25}R$ we have

$$\nabla^2 v(r,t) = \sum_{n=1}^{\infty} -\frac{z_n^2}{25}\, a_n(t) J_0\left(\frac{z_n r}{5}\right).$$

We put this into the differential equation for v and get

$$v_t - 3\nabla^2 v = \sum_{n=1}^{\infty} \left(a_n' + \frac{3z_n^2}{25}a_n\right) J_0\left(\frac{z_n r}{5}\right) = 0.$$

All of the coefficients in this last series must be zero, which gives first-order ordinary differential equations for the $a_n(t)$'s, and we conclude that

$$a_n(t) = c_n e^{-3z_n^2 t/25}$$

and so

$$v(r,t) = \sum_{n=1}^{\infty} c_n e^{-3z_n^2 t/25} J_0\left(\frac{z_n r}{5}\right).$$

Now we have to use the initial conditions for v to find the constants c_n:

$$v(r,0) = \sum_{n=1}^{\infty} c_n J_0\left(\frac{z_n r}{5}\right) = -\tfrac{1}{48}r^4 + \tfrac{25}{12}r^2 - \tfrac{945}{16}.$$

We know how to do this:

$$c_n = \frac{\left\langle -\dfrac{1}{48}r^4 + \dfrac{25}{12}r^2 - \dfrac{945}{16}\,,\, J_0\left(\dfrac{z_n r}{5}\right)\right\rangle}{\left\langle J_0\left(\dfrac{z_n r}{5}\right)\,,\, J_0\left(\dfrac{z_n r}{5}\right)\right\rangle}$$

where the inner product is given as usual in polar coordinates by

$$\langle f(r)\,,\, g(r)\rangle = \int_0^5 f(r)g(r)\, r\, dr.$$

We defer the calculation of the integrals to avoid losing the track of the story (the calculations are given below), and simply report that

$$c_n = -\frac{1}{J_1(z_n)}\left(\frac{40}{z_n} + \frac{5000}{3z_n^5}\right).$$

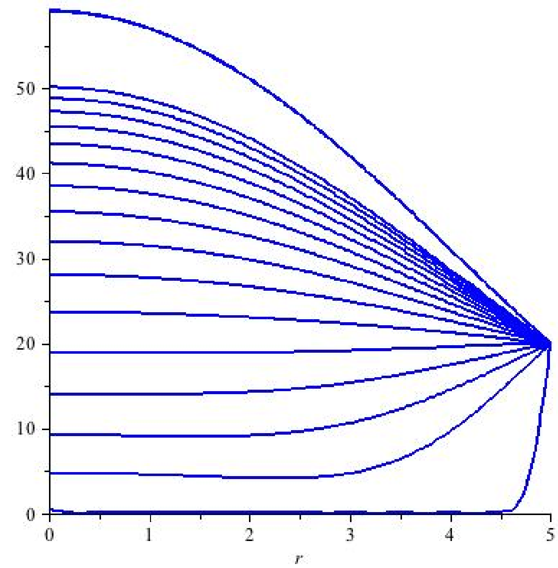
Then we can conclude that

$$v(r,t) + \sum_{n=1}^{\infty} -\frac{1}{J_1(z_n)}\left(\frac{40}{z_n} + \frac{5000}{3z_n^5}\right) e^{-3z_n^2 t/25} J_0\left(\frac{z_n r}{5}\right)$$

and so the solution to our original problem is

$$u(r,t) = \frac{1}{48}r^4 - \frac{25}{12}r^2 + \frac{945}{16} - \sum_{n=1}^{\infty} \frac{1}{J_1(z_n)}\left(\frac{40}{z_n} + \frac{5000}{3z_n^5}\right) e^{-3z_n^2 t/25} J_0\left(\frac{z_n r}{5}\right)$$

Here are graphs of the temperature as a function of the radius for t between 0 and 3 at time steps of 0.2, together with the equilibrium solution they are approaching:



We still have to compute the inner product integrals that produce the coefficients c_n. There are four integrals to do. In the first three, we will repeatedly use the formula

$$\frac{d}{dx}(x^n J_n(x)) = x^n J_{n-1}(x) \quad \text{or} \int x^n J_{n-1}(x)\,dx = x^n J_n(x) + C$$

with various values of n. We'll also be integrating by parts with $dv = x^n J_{n-1}(x)\,dx$ and u equal to whatever powers of x remain unaccounted for in dv (like that problem in the midterm).

Start with the substitution $x = z_n r/5$ (so $r = 5x/z_n$ and $dr = 5dx/z_n$) and compute:

$$\begin{aligned}
\left\langle r^4\,,\ J_0\left(\frac{z_n r}{5}\right)\right\rangle &= \int_0^5 r^5 J_0\left(\frac{z_n r}{5}\right)\,dr = \left(\frac{5}{z_n}\right)^6 \int_0^{z_n} x^5 J_0(x)\,dx \\
&= \left(\frac{5}{z_n}\right)^6 \left[x^5 J_1(x)\Big|_0^{z_n} - \int_0^{z_n} 4x^4 J_1(x)\,dx \right] \\
&= \frac{5^6}{z_n} J_1(z_n) - 4\left(\frac{5}{z_n}\right)^6 \left[x^4 J_2(x)\Big|_0^{z_n} - \int_0^{z_n} 2x^3 J_2(x)\,dx \right] \\
&= \frac{5^6}{z_n} J_1(z_n) - \frac{4\cdot 5^6}{z_n^2} J_2(z_n) + 8\left(\frac{5}{z_n}\right)^6 x^3 J_3(x)\Big|_0^{z_n} \\
&= \frac{5^6}{z_n} J_1(z_n) - \frac{4\cdot 5^6}{z_n^2} J_2(z_n) + \frac{8\cdot 5^6}{z_n^3} J_3(z_n)
\end{aligned}$$

where we integrated by parts with $u = x^4$ and $dv = xJ_0(x)\,dx$ to go from the first to the second line, and we integrated by parts with $u = x^2$ and $dv = x^2 J_1(x)\,dx$ to go from the second to the third line.

Now we can use the Bessel function identity $\frac{2n}{x}J_n(x) = J_{n-1}(x) + J_{n+1}(x)$ (and remember that $J_0(z_n) = 0$ by definition of z_n) with $n = 1$ to get $J_2(z_n) = \frac{2}{z_n}J_1(x)$ and first with $n = 2$ and then with $n = 1$ to get

$$J_3(z_n) = \frac{4}{z_n}J_2(z_n) - J_1(z_n) = \left(\frac{8}{z_n^2} - 1\right)J_1(x).$$

Therefore

$$\begin{aligned}\left\langle r^4\,,\ J_0\left(\frac{z_n r}{5}\right)\right\rangle &= \frac{5^6}{z_n}J_1(z_n) - \frac{4\cdot 5^6}{z_n^2}\frac{2}{z_n}J_1(z_n) + \frac{8\cdot 5^6}{z_n^3}\left(\frac{8}{z_n^2} - 1\right)J_1(x)\\ &= 5^6 J_1(z_n)\left(\frac{1}{z_n} - \frac{16}{z_n^3} + \frac{64}{z_n^5}\right).\end{aligned}$$

In a similar way, we calculate

$$\begin{aligned}\left\langle r^2\,,\ J_0\left(\frac{z_n r}{5}\right)\right\rangle &= \int_0^5 r^3 J_0\left(\frac{z_n r}{5}\right)\,dr = \left(\frac{5}{z_n}\right)^4\int_0^{z_n} x^3 J_0(x)\,dx\\ &= \left(\frac{5}{z_n}\right)^4\left[x^3 J_1(x)\Big|_0^{z_n} - \int_0^{z_n} 2x^2 J_1(x)\,dx\right]\\ &= \frac{5^4}{z_n}J_1(z_n) - 2\left(\frac{5}{z_n}\right)^4 x^2 J_2(x)\Big|_0^{z_n}\\ &= \frac{5^4}{z_n}J_1(z_n) - \frac{2\cdot 5^4}{z_n^2}J_2(z_n)\\ &= \frac{5^4}{z_n}J_1(z_n) - \frac{2\cdot 5^4}{z_n^2}\frac{2}{z_n}J_1(z_n)\\ &= 5^4 J_1(z_n)\left(\frac{1}{z_n} - \frac{4}{z_n^3}\right)\end{aligned}$$

The next one is easy:

$$\left\langle 1\,,\ J_0\left(\frac{z_n r}{5}\right)\right\rangle = \int_0^5 r J_0\left(\frac{z_n r}{5}\right)\,dr = \left(\frac{5}{z_n}\right)^2\int_0^{z_n} xJ_0(x)\,dx = \frac{5^2}{z_n}J_1(z_n).$$

And finally, we use the substitution $x = r/5$ (or $r = 5x$ and $dr = 5\,dx$ together with the orthogonality relation

$$\int_0^1 x(J_m(z_{mn}x))^2\,dx = \tfrac{1}{2}(J_{m+1}(z_{mn}))^2$$

to compute the inner product

$$\left\langle J_0\left(\frac{z_n r}{5}\right)\,,\, J_0\left(\frac{z_n r}{5}\right)\right\rangle = \int_0^5 r(J_0\left(\frac{z_n r}{5}\right))^2\,dr = 5^2\int_0^1 x(J_0(z_n x))^2\,dx = \frac{5^2}{2}(J_1(z_n))^2.$$

We put these all together to compute

$$\begin{aligned}
c_n &= \frac{\left\langle -\frac{1}{48}r^4 + \frac{25}{12}r^2 - \frac{945}{16}\,,\, J_0\left(\frac{z_n r}{5}\right)\right\rangle}{\left\langle J_0\left(\frac{z_n r}{5}\right), J_0\left(\frac{z_n r}{5}\right)\right\rangle} \\
&= \frac{2}{5^2(J_1(z_n))^2}\left(-\frac{1}{48}\left\langle r^4\,,\, J_0\left(\frac{z_n r}{5}\right)\right\rangle + \frac{25}{12}\left\langle r^2\,,\, J_0\left(\frac{z_n r}{5}\right)\right\rangle - \frac{945}{16}\left\langle 1\,,\, J_0\left(\frac{z_n r}{5}\right)\right\rangle\right) \\
&= \frac{1}{J_1(z_n)}\left[-\frac{5^4}{24}\left(\frac{1}{z_n} - \frac{16}{z_n^3} + \frac{64}{z_n^5}\right) + \frac{25\cdot 5^2}{6}\left(\frac{1}{z_n} - \frac{4}{z_n^3}\right) - \frac{945}{8}\frac{1}{z_n}\right] \\
&= \frac{1}{J_1(z_n)}\left(-\frac{40}{z_n} - \frac{5000}{3z_n^5}\right).
\end{aligned}$$

That's it.