1. Let 

\[ f(x) = (x - 1)^2 \quad \text{for } 0 \leq x \leq 1 \]

(a) Compute the Fourier cosine series of \( f(x) \).

We have \( f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\pi x \) where

\[
a_0 = \int_0^1 (x - 1)^2 \, dx = \left. \frac{(x - 1)^3}{3} \right|_0^1 = \frac{1}{3}
\]

and

\[
a_n = 2 \int_0^1 (x - 1)^2 \cos n\pi x \, dx = 2 \left[ \frac{(x - 1)^2}{n\pi} \sin n\pi x + \frac{2(x - 1)}{n^2\pi^2} \cos n\pi x - \frac{2}{n^3\pi^3} \sin n\pi x \right]_0^1 = \frac{4}{n^2\pi^2}
\]

Therefore

\[ (x - 1)^2 = \frac{1}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2} \cos n\pi x \quad \text{for } 0 \leq x \leq 1. \]

(b) Draw a careful graph of the function to which your series converges for \(-4 \leq x \leq 4\).

(c) Substitute appropriate numbers into your series and calculate the values of

\[ \sum_{n=1}^{\infty} \frac{1}{n^2} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}. \]
Put $x = 0$ in the series and get

$$1 = \frac{1}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2 \pi^2}$$

Subtract $\frac{1}{3}$ from both sides and then multiply both sides by $\pi^2/4$ and get

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

Next, put $x = 1$ into the series and get

$$0 = \frac{1}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2 \pi^2} (-1)^n.$$ 

Subtract the series to the left side of the equation and multiply both sides by $\pi^2/4$ and get

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}.$$ 

(d) Do not compute the Fourier sine series of $f(x)$, but draw a careful graph of the function to which the Fourier sine series of $f(x)$ converges for $-4 \leq x \leq 4$.

This is the sum of the first fifty or so terms — where the vertical line segments are, there should be dots on the $x$-axis.

2. Solve the boundary-value problem for the Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

on the square $0 \leq x \leq 2$, $0 \leq y \leq 2$ with boundary values

$$u(x, 0) = 1 \quad \text{and} \quad u(x, 2) = 0 \quad \text{for} \quad 0 \leq x \leq 2$$
and

\[ u_x(0, y) = 0 \quad \text{and} \quad u_x(2, y) = 0 \quad \text{for} \quad 0 \leq y \leq 2. \]

(Hint: This problem is a gift: There’s an easy way to solve this problem — so draw a picture and think about it before you commit Fourier series!)

Because \( u_x = 0 \) on both vertical sides, and since \( u \) is constant on the bottom and on the top, the solution must be a function of \( y \) alone. A harmonic function of one variable is linear, so we can calculate that \( u(x, y) = 1 - \frac{1}{2}y \) is the solution.

If you must use Fourier series, then note that the separated version (letting \( u(x, y) = X(x)Y(y) \), get

\[ \frac{X''}{X} = -\frac{Y''}{Y} = -\lambda \]

where we use \(-\lambda \) because \( X'(0) = X'(2) = 0 \), so \( X = \cos(n\pi x/2) \) for \( n = 0, 1, 2, \ldots \) (don’t forget \( n = 0! \)) and \( \lambda = n^2\pi^2/4 \).

The corresponding \( Y \)'s satisfy

\[ Y'' - \frac{n^2\pi^2}{4} Y = 0 \quad \text{and} \quad Y(2) = 0 \]

so \( Y = a_0(2 - y) \) for \( n = 0 \) and \( Y = a_n \sinh(n\pi(2 - y)/2) \) for other values of \( n \). So far we have

\[ u(x, y) = a_0(2 - y) + \sum_{n=1}^{\infty} a_n \cos \left( \frac{n\pi x}{2} \right) \sinh \left( \frac{n\pi(2 - y)}{2} \right). \]

Now put \( y = 0 \) and get

\[ u(x, 0) = 1 = a_0(2) + \sum_{n=1}^{\infty} a_n \sinh(n\pi \cos \left( \frac{n\pi x}{2} \right). \]

But the cosine series for the function 1 is just 1, so all the \( a_n = 0 \) for \( n \geq 1 \) and \( a_0 = \frac{1}{2} \). Therefore

\[ u(x, t) = \frac{1}{2}(2 - y) \]

in agreement with what we got before.

\[ \text{3. (a) Show that} \int_0^1 r^4 J_1(z_{1m} r) \, dr = \frac{8 - z_{1m}^2}{z_{1m}^3} J_0(z_{1m}) \]

where as usual \( z_{1m} \) is the \( m \)th positive zero of the Bessel function \( J_1(x) \).

(Hint: This problem is not a gift: You will have to integrate by parts, and use at least two of the Bessel function identities.)
First, the substitution $x = z_1 m r$ (so $dx = z_1 m dr$) gives:

$$\int_0^1 r^4 J_1(z_1 m r) \, dr = \frac{1}{z_1 m} \int_0^{z_1 m} x^4 J_1(x) \, dx$$

and then

$$\int_0^{z_1 m} x^4 J_1(x) \, dx = \int_0^{z_1 m} x^2 \cdot x^2 J_1(x) \, dx$$

$$= x^4 J_2(x) \bigg|_0^{z_1 m} - \int_0^{z_1 m} 2x^3 J_2(x) \, dx$$

$$= z_1 m^2 J_2(z_1 m) - 2 \int_0^{z_1 m} 3x^2 J_3(x) \, dx$$

$$= z_1 m^2 J_2(z_1 m) - 2 z_1 m J_3(z_1 m)$$

where we integrated by parts to go from the first to the second line and used identity (1) twice (once to integrate $x^2 J_1(x)$ and once to integrate $x^3 J_2(x)$).

Next, identity (6) with $n = 1$ says (remembering that $J_1(z_1 m) = 0$)

$$\frac{2}{x} J_1(x) = J_0(x) + J_2(x) \quad \text{so} \quad 0 = J_0(z_1 m) + J_2(z_1 m)$$

and with $n = 2$ it says

$$\frac{4}{x} J_2(x) = J_1(x) + J_3(x) \quad \text{so} \quad \frac{4}{z_1 m} J_2(z_1 m) = J_3(z_1 m),$$

or, substituting the first of these into the second, $J_3(z_1 m) = -4 J_0(z_1 m) / z_1 m$. Thus

$$\int_0^{z_1 m} x^4 J_1(x) \, dx = -z_1 m^2 J_0(z_1 m) + 8 z_1 m J_0(z_1 m)$$

and so

$$\int_0^1 r^4 J_1(z_1 m r) \, dr = \frac{8 - z_1 m^2}{z_1 m} J_0(z_1 m).$$

(b) Using the result of part (a), even if you couldn’t prove it, compute the Fourier-Bessel series of the form

$$\sum_{m=1}^{\infty} a_m J_1(z_1 m r)$$

for the function $f(r) = r^3$ for $0 \leq r \leq 1$. 
The coefficients are
\[
a_m = \frac{\langle r^3, J_1(z_{1m}r) \rangle}{\langle J_1(z_{1m}r), J_1(z_{1m}r) \rangle} = \int_0^1 r^3 J_1(z_{1m}r) r \, dr = \frac{8 - z_{1m}^2 J_0(z_{1m})}{z_{1m}^2 J_2(z_{1m})^2}
\]
and use the fact that \( J_2(z_{1m}) = -J_0(z_{1m}) \) from part (a) to simplify this to
\[
a_m = \frac{8 - z_{1m}^2}{z_{1m}^2 J_0(z_{1m})}.
\]
Thus
\[
r^3 = \sum_{m=1}^{\infty} \frac{8 - z_{1m}^2}{z_{1m}^2 J_0(z_{1m})} J_1(z_{1m}r).
\]

(c) Use this series to solve the initial-boundary value problem for the wave equation:
\[
u_{tt} = 4\nabla^2 u
\]
on the unit disk \( (x^2 + y^2 \leq 1 \), or equivalently \( 0 \leq r \leq 1 \) \) with initial conditions
\[
u(r, \theta, 0) = r^3 \cos \theta \quad \text{and} \quad \nu_t(r, \theta, 0) = 0 \quad \text{for} \quad 0 \leq r \leq 1, \ 0 \leq \theta \leq 2\pi
\]
and zero boundary values, \( u(1, \theta, t) = 0 \) for \( 0 \leq \theta \leq 2\pi, \ t > 0 \).

In polar coordinates,
\[
u_{tt} = 4 \left( \frac{1}{r} (ru_r)_r + \frac{1}{r^2} u_{\theta\theta} \right)
\]
Separating variables \( \left( u(r, \theta, t) = R(r)\Theta(\theta)T(t) \right) \) gives
\[
\frac{T''}{4T} = \frac{(rR')'}{rrR} + \frac{\Theta''}{r^2\Theta} = -\lambda
\]
so
\[
T = c_1 + c_2 t \quad \text{or} \quad T = c_1 \cos(2\sqrt{\lambda}t) + c_2 \sin(2\sqrt{\lambda}t)
\]
depending on whether \( \lambda \) is equal to or greater than zero, respectively. Since \( u_t(r, \theta, 0) = 0 \), we know that all the \( c_2 \)'s will be zero, so we need only keep the constant and cosine terms.

Next, multiply the right separated equation by \( r^2 \) and rearrange to get
\[
\frac{r^2 R'' + r R'}{R} + \lambda r^2 = -\frac{\Theta''}{\Theta} = \mu
\]
and since \( \Theta \) must be periodic, we have that \( \mu = n^2 \) and
\[
\Theta = a_n \cos n\theta + b_n \sin n\theta.
\]
The \( R \) equation becomes
\[
r^2 R'' + r R' + (\lambda r^2 - n^2) R = 0
\]
which is Bessel’s equation of order \( n \). Since we need \( R(1) = 0 \) from the boundary condition (and \( R(0) \) is bounded) we have
\[
R(r) = J_n(z_{nm} r) \quad \text{and} \quad \lambda = z_{nm}^2
\]
(or \( R = 1 \) for \( \lambda = 0 \)). Thus, the solution is
\[
u(r, \theta, t) = a_0 + \sum_{m=1}^{\infty} a_{nm} J_0(z_{nm} r) \cos(2z_{nm} t) [a_{nm} \cos n\theta + b_{nm} \sin n\theta].
\]
But the initial condition \( u(r, \theta, 0) = r^3 \cos \theta \) tells us that all the \( b_{nm} \)'s are zero and all the \( a_{nm} \)'s are zero except when \( n = 1 \). So the solution reduces to
\[
u(r, \theta, t) = \sum_{m=1}^{\infty} a_{1m} J_1(z_{1m} r) \cos(2z_{1m} t).
\]
Applying the initial condition and the solution to part (b) gives the final answer:
\[
u(r, \theta, t) = \sum_{m=1}^{\infty} \frac{8 - z_{1m}^2}{z_{1m}^2 J_0(z_{1m})} J_1(z_{1m} r) \cos(2z_{1m} t).
\]

4. Solve the inhomogeneous wave equation
\[
u_{tt} = u_{xx} + \sin x, \quad 0 < x < \pi, \quad t > 0
\]
where \( u(x, 0) = \sin 2x, \ u_t(x, 0) = 0, \ u(0, t) = 0 \) and \( u(\pi, t) = 0 \).

(Hint: The eigenfunctions of \( X'' + \lambda X = 0 \) on \([0, \pi]\) with \( X(0) = X(\pi) = 0 \) are \( X = \sin nx \) and \( \lambda = n^2 \) for \( n = 1, 2, \ldots \)). So it makes sense to try a solution of the form
\[
u(x, t) = \sum_{n=1}^{\infty} a_n(t) \sin nx.
\]
Put this into the equation and determine what the functions \( a_n(t) \) have to be. You will get ordinary differential equations for the \( a_n(t) \) from the inhomogeneous wave equation,
and initial conditions for them from the problem’s initial conditions. This should not require a lot of computation. Most of the $a_n$’s will be zero, so that the answer is a finite sum, not a series.)

From

$$u(x, t) = \sum_{n=1}^{\infty} a_n(t) \sin nx$$

we get

$$u_{tt} = \sum_{n=1}^{\infty} a_n''(t) \sin nx \quad \text{and} \quad u_{xx} = \sum_{n=1}^{\infty} -n^2 a_n(t) \sin nx.$$ 

Putting these into the equation gives

$$\sum_{n=1}^{\infty} (a_n''(t) + n^2 a_n(t)) \sin nx = \sin x.$$ 

Therefore

$$a_1'' + a_1 = 1 \quad \text{and} \quad a_n'' + n^2 a_n = 0 \quad \text{for} \quad n \geq 2.$$ 

The general solutions of these equations are

$$a_1(t) = 1 + c_1 \cos t + d_1 \sin t \quad \text{and} \quad a_n(t) = c_n \cos nt + d_n \sin nt$$

respectively. Therefore,

$$u(x, t) = \sin x + \sum_{n=1}^{\infty} \sin nx[c_n \cos nt + d_n \sin nt].$$

Now use the initial condition:

$$u(x, 0) = \sin 2x = \sin x + \sum_{n=1}^{\infty} c_n \sin nx$$

to learn that $c_1 = -1$, $c_2 = 1$ and $c_n = 0$ for $n \geq 3$. Likewise,

$$u_t(x, 0) = 0 = \sum_{n=1}^{\infty} nd_n \sin nx$$

tells us that $d_n = 0$ for all $n$. So the solution is

$$u(x, t) = \sin x - \sin x \cos t + \sin 2x \cos 2t.$$
5. Find the steady-state temperature distribution \( u(r, \theta, z) \) on the solid cylinder with radius 2 (so \( 0 \leq r \leq 2 \)) and height 3 (so \( 0 \leq z \leq 3 \)) if the temperature at the ends is held at zero, so

\[
u(r, \theta, 0) = 0 \quad \text{and} \quad u(r, \theta, 3) = 0
\]

and the temperature on the lateral side is 1, so \( u(2, \theta, z) = 1 \).

This is the case of “Part III” in the notes on the Laplacian on the cylinder. Since the top and bottom are held at zero temperature, we have that \( u(r, \theta, z) \) is what we called there \( u_3(r, \theta, z) \). So

\[
u(r, \theta, z) = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} I_n \left( \frac{m\pi r}{H} \right) \sin \left( \frac{m\pi z}{H} \right) \left[ e_{nm} \cos n\theta + f_{nm} \sin n\theta \right],
\]

with \( H = 3 \), so

\[
u(r, \theta, z) = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} I_n \left( \frac{m\pi r}{3} \right) \sin \left( \frac{m\pi z}{3} \right) \left[ e_{nm} \cos n\theta + f_{nm} \sin n\theta \right],
\]

We have to choose the coefficients \( e_{nm} \) and \( f_{nm} \) so that

\[
u(2, \theta, z) = 1 = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} I_n \left( \frac{2m\pi}{3} \right) \sin \left( \frac{m\pi z}{3} \right) \left[ e_{nm} \cos n\theta + f_{nm} \sin n\theta \right],
\]

Since the boundary data is a constant, we know immediately that \( e_{nm} = 0 \) for all \( m \geq 1 \) and \( n \) and \( f_{nm} = 0 \) for all \( m \) and all \( n \). So we can simplify things to finding the coefficients \( e_{0m} \) so that

\[
1 = \sum_{m=1}^{\infty} e_{0m} I_0 \left( \frac{2m\pi}{3} \right) \sin \left( \frac{m\pi z}{3} \right)
\]

for \( 0 \leq z \leq 3 \). This is an ordinary Fourier sine series, so we know that

\[
e_{0m} = \frac{1}{I_0 \left( \frac{2m\pi}{3} \right)} \frac{2}{3} \int_0^3 \sin \left( \frac{m\pi z}{3} \right) \, dz
\]

\[
= \frac{1}{I_0 \left( \frac{2m\pi}{3} \right)} \frac{2}{3} \left( - \frac{3}{n\pi} \cos \frac{m\pi z}{3} \right) \bigg|_0^3
\]

\[
= - \frac{2}{m\pi I_0 \left( \frac{2m\pi}{3} \right)} \left[ (-1)^m - 1 \right]
\]

\[
= \frac{1}{I_0 \left( \frac{2m\pi}{3} \right)} \begin{cases}
  0 & n \text{ even} \\
  \frac{4}{n\pi} & n \text{ odd}
\end{cases}
\]
So the solution of the problem is

\[ u(r, \theta, z) = \sum_{k=0}^{\infty} \frac{4}{(2k+1)\pi} I_0 \left( \frac{(4k+2)\pi r}{3} \right) \sin \left( \frac{(2k+1)\pi z}{3} \right) \]

6. Find the first three non-zero terms of the power series (centered at zero) for the solution of the equation

\[ x^2 \frac{d^2 y}{dx^2} + (x^2 - 2)y = 0 \]

that is bounded at \( x = 0 \). Does the solution oscillate (like sines and cosines and \( J_n(x) \)) or does it increase steadily to infinity (like \( e^x \) and hyperbolic sines and cosines and \( I_n(x) \))?

The point \( x = 0 \) is a regular singular point for this equation, so we must assume that

\[ y = \sum_{n=0}^{\infty} a_n x^{n+s} \]

with \( a_0 \neq 0 \) and discover what values of \( s \) work. If we substitute this assumption into the differential equation, we get:

\[ \sum_{n=0}^{\infty} (n+s)(n+s-1)a_n x^{n+s} + \sum_{n=0}^{\infty} a_n x^{n+s+2} - \sum_{n=0}^{\infty} 2a_n x^{n+s} = 0. \]

Put \( m = n + 2 \) in the middle term (so \( n = m - 2 \) and \( m \) will go from 2 to \( \infty \)). Then we can rewrite the whole thing as a single sum from 2 to \( \infty \) with the \( n = 0 \) and \( n = 1 \) terms from the first and third sums separated out as follows:

\[ [s(s-1) - 2]a_0 x^s + [(s+1)s - 2]a_1 x^{1+s} + \sum_{n=2}^{\infty} [((n+s)(n+s-1) - 2) a_n + a_{n-2}] x^{n+s} = 0 \]

In order for \( a_0 \neq 0 \) we must have \( s(s-1) - 2 = s^2 - s - 2 = (s-2)(s+1) = 0 \), which gives \( s = 2 \) or \( s = -1 \). The solution with \( s = -1 \) will start with \( 1/x \) and so cannot be bounded at \( x = 0 \), so we look only at the \( s = 2 \) solution, which will start with \( x^2 \).

Next, the \( x^{1+s} \) term has coefficient \([s+1)s - 2]a_1\), which is \( 4a_1 \) for \( s = 2 \). This tells us that \( a_1 = 0 \). And from the summation, we get the recurrence relation (using \( s = 2 \) straightaway)

\[ [(n+2)(n+1) - 2]a_n + a_{n-2} = 0 \]

or

\[ a_n = - \frac{a_{n-2}}{n(n+3)}. \]
Since $a_1 = 0$, we have $a_n = 0$ for all odd values of $n$. For the first few even values of $n$ we have

\[
a_2 = -\frac{a_0}{2 \cdot 5}
\]
\[
a_4 = -\frac{a_2}{4 \cdot 7} = -\frac{a_0}{(2 \cdot 4)(5 \cdot 7)}
\]
\[
a_6 = -\frac{a_4}{6 \cdot 9} = -\frac{a_0}{(2 \cdot 4 \cdot 6)(5 \cdot 7 \cdot 9)}
\]

Therefore, the first three non-zero terms are:

\[
a_0 \left( x^2 - \frac{1}{10} x^4 + \frac{1}{280} x^6 + \cdots \right)
\]

By the ratio test, the series will converge for all $x$. And since we can rewrite the differential equation as

\[
y'' + \left(1 - \frac{2}{x^2}\right)y = 0,
\]

from the Sturm comparison theorem (remember that??) we see that the solution oscillates and has zeroes separated by a little more than $\pi$, and the separation approaches $\pi$ for large values of $x$, since $1 - \frac{2}{x^2}$ approaches 1 from below.