1. Fluid is flowing along a thin tube of length 3 meters, so that one end of the tube is at \( x = 0 \) and the other is at \( x = 3 \). Let \( \rho(x,t) \) be the linear density of fluid in grams/centimeter (we’ll assume the cross-sectional area of the tube is constant and taken into account in this quantity), and that the flux from left to right at the point \( x \) at time \( t \) is \( \varphi(x,t) \) grams per second.

Suppose the density at time \( t = 0 \) is \( \rho(x,0) = 3 + x \) and the flux is measured and shown to be steady (independent of time) and is \( \varphi(x,t) = 1 - x \). What is \( \rho(x,2) \), i.e., what is the density function of the fluid at time \( t = 2 \)?

(A) \( 3 + x \)  
(B) \( 5 + x \)  
(C) \( 5 - x \)  
(D) \( 3 + 3x - \frac{1}{2}x^2 \)  
(E) \( 2 - 2x \)  
(F) \( 3 - 2x \)

2. Heat is being applied along a thin wire so that its temperature \( u(x,t) \) satisfies the inhomogeneous heat equation

\[
\frac{\partial u}{\partial t} = \frac{1}{4} \frac{\partial^2 u}{\partial x^2} + 4 - x^2 \quad \text{for } 0 < x < 2 \text{ and } t > 0
\]

If the boundary conditions are

\[
\begin{align*}
    u(0,t) &= 0 \\
    u_x(2,t) &= 0
\end{align*}
\]

then what is

\[
\lim_{t \to \infty} u(2,t),
\]

i.e., what is the equilibrium temperature at the insulated end of the wire?

(A) 0  
(B) \( 1/2 \)  
(C) 16  
(D) 4  
(E) \( 8/3 \)  
(F) 6
3. Let $u(x, t)$ be a solution of the heat equation for a wire of length $L$ with insulated ends (so $u_x(0, t) = u_x(L, t) = 0$). Which of the following is (are) true? Give reasons for your answers (either mathematical or physical reasoning is acceptable).

1. $\lim_{t \to \infty} u(x, t) = 0$ for all $x$ between 0 and $L$.

2. The value of $\int_0^L u(x, t) \, dx$ is independent of $t$.

3. The value of $\int_0^L u_x(x, t) \, dx$ approaches zero as $t \to \infty$.

(A) (1) only \hspace{1cm} (B) (2) only \hspace{1cm} (C) (3) only

(D) (1) and (2) only \hspace{1cm} (E) (2) and (3) only \hspace{1cm} (F) (1) and (3) only

4. Let $u(x, t)$ be the solution of the problem:

$$u_t = u_{xx} \quad \text{for} \quad 0 < x < 4 \quad \text{and} \quad t > 0$$

$$u(0, t) = 0 \quad \text{and} \quad u(4, t) = 0$$

$$u(x, 0) = 3 \sin 8\pi x$$

What is $u(\frac{3}{10}, 3)$? (This problem should be easy, except maybe the evaluation part)

(A) $3e^{-\pi^2}$ \hspace{1cm} (B) $3e^{-6\pi^2}$ \hspace{1cm} (C) $3e^{-192\pi^2}$

(D) $-3e^{-\pi^2}$ \hspace{1cm} (E) $-3e^{-6\pi^2}$ \hspace{1cm} (F) $-3e^{-192\pi^2}$
5. If, for $0 \leq x \leq 5$, we have

$$\sum_{n=1}^{\infty} b_n \sin \left( \frac{n\pi x}{5} \right) = 5x - x^2,$$

then $b_3 = \begin{cases} 
(A) \frac{2}{9\pi^2} & (B) \frac{20}{9\pi^2} & (C) \frac{200}{27\pi^3} \\
(D) \frac{1000}{27\pi^3} & (E) \frac{100}{81\pi^4} & (F) \frac{2000}{81\pi^4}
\end{cases}$

Bonus problem: Substitute $x = 5/2$ into the series and the function, then think, do some algebra, and so find the value of

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n + 1)^3}.$$ 

6. Suppose the flux at every point of the outer circle of an annulus with inner radius $R_1 = 2$ and outer radius $R_2 = 5$ points directly out of the annulus and has magnitude 4. Also, suppose that at every point of the inner circle of the annulus the flux points directly into the annulus and has the same magnitude all around the circle. What must the magnitude of this latter flux be so that the temperature of the annulus will be at equilibrium? (In other words, so that there is a solution of the Laplace equation with these flux values.)

\begin{align*}
(A) & -4 & (B) & 0 & (C) & 4 \\
(D) & -10 & (E) & 10 & (F) & 8
\end{align*}
7. The eigenvalue problem

\[ X'' + 4X' = \lambda X \quad X(0) = X(\pi) = 0 \]

for the function \(X(x)\) on the interval \([0, \pi]\) arises from separation of variables in the partial differential equation \(u_t = u_{xx} + 4u_x\) with boundary conditions \(u(0, t) = u(\pi, t) = 0\). What are the eigenvalues? (In other words, for what values of \(\lambda\) are there non-zero solutions of \(X'' + 4X' = \lambda X\) which satisfy \(X(0) = X(\pi) = 0\)?)

(A) \(-1, -2, -3, \ldots\)  
(B) \(-1, -4, -9, \ldots\)  
(C) \(-3, -6, -11, \ldots\)  
(D) \(-5, -8, -13, \ldots\)  
(E) \(-9, -12, -17, \ldots\)  
(F) \(-1, -8, -27, \ldots\)

8. Solve the initial/boundary value problem

\[ u_t = 2u_{xx} \quad \text{for } 0 < x < 2 \text{ and } t > 0 \]

\[ u(0, t) = u(2, t) = 0 \]

\[ u(x, 0) = \begin{cases} 
  x & 0 \leq x \leq 1 \\
  2 - x & 1 \leq x \leq 2 
\end{cases} \]

(A) \(\sum_{n=0}^{\infty} \frac{(-1)^n}{\pi^2(2n+1)^2} e^{-\frac{(2n+1)^2 \pi^2 t}{4}} \sin \left( \frac{(2n+1)\pi x}{2} \right)\)

(B) \(\sum_{n=0}^{\infty} \frac{(-1)^n}{\pi^2(2n+1)^2} e^{-\frac{(2n+1)^2 \pi^2 t}{4}} \sin \left( \frac{(2n+1)\pi x}{2} \right)\)

(C) \(\sum_{n=0}^{\infty} \frac{(-1)^n}{\pi^2(2n+1)^2} e^{-\frac{(2n+1)^2 \pi^2 t}{2}} \sin ((2n + 1)\pi x)\)

(D) \(\sum_{n=0}^{\infty} \frac{(-1)^n}{\pi^2(2n+1)^2} e^{-\frac{(2n+1)^2 \pi^2 t}{4}} \sin \left( \frac{2n\pi x}{2} \right)\)

(E) \(\sum_{n=0}^{\infty} \frac{(-1)^n}{\pi^2(2n)^2} e^{-\frac{(2n)^2 \pi^2 t}{2}} \sin \left( \frac{2n\pi x}{2} \right)\)

(F) \(\sum_{n=0}^{\infty} \frac{(-1)^n}{\pi^2(2n)^2} e^{-\frac{(2n)^2 \pi^2 t}{4}} \sin \left( \frac{2n\pi x}{2} \right)\)
9. Solve the boundary value problem

\[ u_{xx} + u_{yy} = 0 \]

for \((x, y)\) in the quarter-disk \(x > 0, y > 0\) and \(x^2 + y^2 < 1\), and

\[ u(x, 0) = 0 \quad u(0, y) = 0 \quad u(x, \sqrt{1 - x^2}) = 1 \]

(a switch to polar coordinates will help!).

(A) \( \sum_{n=0}^{\infty} \frac{(-1)^n 4}{(2n + 1)^2 \pi^2} r^{2n+1} \sin(2(2n + 1)\theta) \)

(B) \( \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n + 1)^2 \pi^2} r^{2n+1} \sin(2(2n + 1)\theta) \)

(C) \( \sum_{n=0}^{\infty} \frac{4}{(2n + 1)^2 \pi^2} r^{2n+1} \sin(2(2n + 1)\theta) \)

(D) \( \sum_{n=0}^{\infty} \frac{(-1)^n 4}{(2n + 1) \pi} r^{2n+1} \sin(2(2n + 1)\theta) \)

(E) \( \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n + 1) \pi} r^{2n+1} \sin(2(2n + 1)\theta) \)

(F) \( \sum_{n=0}^{\infty} \frac{4}{(2n + 1) \pi} r^{2n+1} \sin(2(2n + 1)\theta) \)