1. Let  
\[ f(x) = \begin{cases} 
1 - x & \text{for } 0 \leq x \leq 1 \\
0 & \text{for } 1 \leq x \leq 2 
\end{cases} \]

(a) Compute the Fourier sine series of \( f(x) \).

The Fourier sine series is 
\[
\sum_{n=1}^{\infty} b_n \sin \left( \frac{n\pi x}{2} \right)
\]

where 
\[
b_n = \frac{2}{2} \int_{0}^{2} f(x) \sin \left( \frac{n\pi x}{2} \right) \, dx = \int_{0}^{1} (1 - x) \sin \left( \frac{n\pi x}{2} \right) \, dx
\]
\[
= -\frac{2}{n\pi} (1 - x) \cos \left( \frac{n\pi x}{2} \right) \bigg|_0^1 - \frac{2}{n\pi} \int_{0}^{1} \cos \left( \frac{n\pi x}{2} \right) \, dx
\]
\[
= \frac{2}{n\pi} - \frac{4}{n^2\pi^2} \sin \left( \frac{n\pi x}{2} \right) \bigg|_0^1
\]
\[
= \begin{cases} 
\frac{2}{n\pi} & \text{if } n \text{ is even} \\
\frac{2}{n\pi} - \frac{4}{n^2\pi^2} & \text{if } n = 4k + 1 \\
\frac{2}{n\pi} + \frac{4}{n^2\pi^2} & \text{rm if } n = 4k + 3
\end{cases}
\]

(b) Draw a careful graph of the function to which your series converges for \(-6 \leq x \leq 6\).

(c) Compute the Fourier cosine series of \( f(x) \).

The Fourier cosine series is 
\[
\sum_{n=0}^{\infty} a_n \cos \left( \frac{n\pi x}{2} \right)
\]
where

$$a_0 = \frac{1}{2} \int_0^2 f(x) \, dx = \frac{1}{2} \int_0^1 1 - x \, dx = \frac{1}{2} \left( x - \frac{x^2}{2} \right) \bigg|_0^1 = \frac{1}{4}$$

and

$$a_n = \frac{2}{2} \int_0^2 f(x) \cos \left( \frac{n\pi x}{2} \right) \, dx = \int_0^1 (1 - x) \cos \left( \frac{n\pi x}{2} \right) \, dx$$

$$= \frac{2}{n\pi} (1 - x) \sin \left( \frac{n\pi x}{2} \right) \bigg|_0^1 + \frac{2}{n\pi} \int_0^1 \sin \left( \frac{n\pi x}{2} \right) \, dx$$

$$= -\frac{4}{n^2\pi^2} \cos \left( \frac{n\pi x}{2} \right) \bigg|_0^1$$

$$= \begin{cases} 
\frac{4}{n^2\pi^2} & \text{if } n \text{ is odd} \\
\frac{8}{n^2\pi^2} & \text{if } n = 4k + 2 \\
0 & \text{rm if } n = 4k
\end{cases}$$

(d) Draw a careful graph of the function to which your series converges for $-6 \leq x \leq 6$. 

2. Say that a function is *oddly odd* if it satisfies both the conditions

$$f(-x) = -f(x), \quad f(L + x) = f(L - x)$$

(a) Show that such a function is periodic with period $4L$. 

We have to show that \( f(x + 4L) = f(x) \) for all \( x \). Well,
\[
\begin{align*}
f(x + 4L) &= f(L + (x + 3L)) & \text{getting ready to use the second condition} \\
&= f(L - (x + 3L)) & \text{by the second condition} \\
&= f(-x - 2L) & \text{duh} \\
&= -f(x + 2L) & \text{by the first condition} \\
&= -f(L + (x + L)) & \text{getting ready to use the second condition again} \\
&= -f(L - (x + L)) & \text{by the second condition} \\
&= -f(-x) \\
&= f(x) & \text{by the first condition}
\end{align*}
\]

(b) Draw the graph of a non-zero oddly odd function for \(-5L \leq x \leq 5L\) (pick one that is interesting but not too interesting, perhaps have the graph consist mostly of line segments). What (if any) kind of symmetry does it have around the line \( x = L \)? …around the line \( x = 2L \)?

![Graph of a function]

In this figure, \( L = 1 \). The function is even around \( x = L \) (i.e., \( f(L + x) = f(L - x) \)) and odd around \( x = 2L \) (i.e., \( f(2L + x) = -f(2L - x) \)).

(c) Show that the Fourier series of an oddly odd function is of the form
\[
f(x) = \sum_{n=1}^{\infty} b_n \sin \left( \frac{(2n-1)\pi x}{2L} \right).
\]
Give a formula for the coefficients \( b_n \).

Since we know the function is odd, we expect only sines. And since it has period \( 4L \), we know that we only have to use its values on the interval \( 0 \leq x \leq 2L \), and the function should have a series expansion of the form
\[
f(x) = \sum_{n=1}^{\infty} b_n \sin \left( \frac{n\pi x}{2L} \right) \quad \text{where} \quad b_n = \frac{1}{L} \int_{0}^{2L} f(x) \sin \left( \frac{n\pi x}{2L} \right) \, dx
\]
But since \( f(2L - x) = f(x) \) (from our symmetry observation in part (b) and the middle of our chain of equalities in part (a)) , we can change variables (let \( u = 2L - x \) so \( du = -dx \) etc) and get

\[
\int_{L}^{2L} f(x) \sin \left( \frac{n\pi x}{2L} \right) \, dx = \int_{0}^{L} f(2L - u) \sin \left( \frac{n\pi(2L - u)}{2L} \right) \, du
\]

\[
= \int_{0}^{L} f(u) \sin \left( n\pi - \frac{n\pi u}{2L} \right) \, du
\]

\[
= - \int_{0}^{L} f(u)(-1)^n \sin \left( \frac{n\pi u}{2L} \right) \, du
\]

Therefore

\[
b_n = \frac{1}{L} \int_{0}^{2L} f(x) \sin (n\pi x 2L) \, dx = \frac{1}{L} \left( \int_{0}^{L} f(x) \sin \left( \frac{n\pi x}{2L} \right) \, dx + \int_{L}^{2L} f(x) \sin \left( \frac{n\pi x}{2L} \right) \, dx \right)
\]

\[
= \frac{1}{L} \left( \int_{0}^{L} f(x) \sin \left( \frac{n\pi x}{2L} \right) \, dx - (-1)^n \int_{0}^{L} f(x) \sin \left( \frac{n\pi x}{2L} \right) \, dx \right)
\]

\[
= \frac{1 - (-1)^n}{L} \int_{0}^{L} f(x) \sin \left( \frac{n\pi x}{2L} \right) \, dx
\]

So we have \( b_n = 0 \) if \( n \) is even, and if \( n \) is odd, say \( n = 2k - 1 \),

\[
b_{2k-1} = \frac{2}{L} \int_{0}^{L} f(x) \sin \left( \frac{(2k-1)\pi x}{2L} \right) \, dx
\]

and so

\[
f(x) = \sum_{k=1}^{\infty} b_{2k-1} \sin \left( \frac{(2k-1)\pi x}{2L} \right)
\]

with these coefficients.

3. Solve the initial-boundary value problem for the wave equation:

\[
u_{tt} = 4u_{xx}, \quad 0 < x < 1, \quad t > 0
\]

where \( u(x,0) = \sin \pi x, \, u_t(x,0) = 0, \, u(0,t) = 0, \, u(1,t) = 1. \)

(Hint: What is the equilibrium solution?)

We have to use the equilibrium solution \( u_{eq}(x) \) in order to satisfy the non-homogeneous boundary condition at \( x = 1 \). Since the equilibrium solution has \( u_{tt} = 0 \), it will be a linear function in \( x \) satisfying \( u_{eq}(0) = 0 \) and \( u_{eq}(1) = 1 \), so \( u_{eq}(x) = x. \)
Now let \( v(x, t) = u(x, t) - u_{eq} = u(x, t) - x \). Then \( v \) satisfies the same differential equation as \( u \), so \( v_t = 4v_{xx} \), but the boundary conditions for \( v \) are \( v(0, t) = v(1, t) = 0 \), and the initial conditions for \( v \) will be \( v(x, 0) = \sin \pi x - x \) and \( v_t(x, 0) = 0 \).

By the usual separation of variables, let \( v(x, t) = X(x)T(t) \), then we’ll have

\[
\frac{T''}{4T} = \frac{X''}{X} = -\lambda
\]

Since \( X(0) = X(1) = 0 \), we will have \( X = \sin n\pi x \) with \( n = 1, 2, 3, \ldots \) and \( \lambda = n^2 \pi^2 \). Then \( T'' + 4n^2 \pi^2 T = 0 \) so \( T = a_n \cos 2n\pi t + b_n \sin 2n\pi t \).

Since \( v_t(x, 0) = 0 \) we know that there are no \( \sin 2n\pi t \) terms in the answer. We thus have

\[
v(x, t) = \sum_{n=1}^{\infty} a_n \sin n\pi x \cos 2n\pi t
\]

where, since \( v(x, 0) = \sin \pi x - x \),

\[
a_1 = 1 - 2 \int_0^1 x \sin \pi x \, dx \quad \text{and} \quad a_n = -2 \int_0^1 x \sin n\pi x \, dx \quad \text{for} \quad n \geq 2.
\]

Since

\[
\int_0^1 x \sin n\pi x \, dx = -\frac{1}{n\pi} x \cos n\pi x \bigg|_0^1 + \int_0^1 \cos n\pi x \, dx = \frac{(-1)^{n+1}}{n\pi},
\]

we have

\[
v(x, t) = \sin \pi x \cos 2\pi t + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\pi} \sin n\pi x \cos 2n\pi t
\]

and so

\[
u(x, t) = u_{eq}(x, t) + v(x, t) = x + \sin \pi x \cos 2\pi t + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\pi} \sin n\pi x \cos 2n\pi t
\]

4. (a) Find the Fourier-Bessel series of the form

\[
\sum_{m=1}^{\infty} a_m J_0(z_0m) r
\]

for the function \( f(r) = 1 \) for \( 0 \leq r \leq 1 \).

From the text or the notes, we know that the coefficients will be of the form

\[
a_m = \frac{\int_0^1 r J_0(z_0m) \, dr}{\int_0^1 (r J_0(z_0m))^2 \, dr}.
\]
From the bottom of page 5 of the notes on the wave equation on the disk we recall that
\[ \int_0^1 x(J_m(z_{nm}x))^2 \, dx = \frac{1}{2} J_{m+1}(z_{nm})^2, \]
so the denominator is \( \frac{1}{2} J_1(z_{0m})^2 \).

As for the numerator, formula (1) from those notes, with \( n = 1 \), says
\[ \frac{d}{dx}(xJ_1(x)) = xJ_0(x). \]
Therefore
\[ \int xJ_0(x) \, dx = xJ_1(x) + C \]
so (making the substitution \( s = z_{0m}r \))
\[ \int_0^1 r J_0(z_{0m}r) \, dr = \int_0^{z_{0m}} \frac{s}{z_{0m}} J_0(s) \, ds = \frac{1}{2} \int_0^{z_{0m}} s J_0(s) \, ds = \frac{J_1(z_{0m})}{z_{0m}}. \]
So we conclude that
\[ 1 = \sum_{m=1}^{\infty} \frac{2}{z_{0m} J_1(z_{0m})} J_0(z_{0m}r) \]
on the interval \( 0 < r < 1 \).

(b) Use this series to solve the initial-boundary value problem for the heat equation:
\[ u_t = 3 \nabla^2 u \]
on the unit disk \( x^2 + y^2 \leq 1 \), or equivalently \( 0 \leq r \leq 1 \) with initial condition
\[ u(r, \theta, 0) = 1 \quad \text{for } 0 \leq r \leq 1, \ 0 \leq \theta \leq 2\pi \]
and zero boundary values, \( u(1, \theta, t) = 0 \) for \( 0 \leq \theta \leq 2\pi, \ t > 0 \).

Since neither the initial nor the boundary condition contain \( \theta \), we know that the solution will be independent of \( \theta \). So we can assume \( u(r, \theta, t) = u(r, t) \) and do \( u = R(r)T(t) \) in the separation of variables. We obtain
\[ \frac{T'}{3T} = \frac{(rR')'}{rR} = -\lambda, \]
so \( T' + 3\lambda T = 0 \) (which will give us \( T = e^{-3\lambda t} \)) and \( rR'' + R' + \lambda rR = 0 \). Multiplying the last equation by \( r \) gives us Bessel’s equation of order zero:
\[ r^2 R'' + rR' + \lambda r^2 R = 0 \]
Therefore $R = J_0(z_{0m}r)$ and $\lambda = z_{0m}^2$. We conclude that

$$u(r, t) = \sum_{m=1}^{\infty} a_m e^{-3z_{0m}^2 t} J_0(z_{0m}r)$$

where by the initial condition we need

$$u(0, t) = \sum_{m=1}^{\infty} a_m J_0(z_{0m}r) = 1.$$ 

But we found these coefficients in part (a), so the solution to the problem is

$$u(r, t) = \sum_{m=1}^{\infty} \frac{2}{z_{0m} J_1(z_{0m})} e^{-3z_{0m}^2 t} J_0(z_{0m}r).$$

5. Find the steady-state temperature distribution on the solid cylinder with radius 3 (so $0 \leq r \leq 3$) and height 5 (so $0 \leq z \leq 5$) if the temperature at the ends is held at zero and the temperature on the lateral side is given by $u(3, \theta, z) = z(5 - z) \sin 3\theta$.

This is the case of “Part III” in the notes on the Laplacian on the cylinder. Since the top and bottom are held at zero temperature, we have that $u(r, \theta, z)$ is what we called there $u_3(r, \theta, z)$. So

$$u(r, \theta, z) = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} I_n \left( \frac{m\pi r}{H} \right) \sin \left( \frac{m\pi z}{H} \right) [e_{nm} \cos n\theta + f_{nm} \sin n\theta],$$

with $H = 5$, so

$$u(r, \theta, z) = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} I_n \left( \frac{m\pi r}{5} \right) \sin \left( \frac{m\pi z}{5} \right) [e_{nm} \cos n\theta + f_{nm} \sin n\theta],$$

We have to choose the coefficients $e_{nm}$ and $f_{nm}$ so that

$$u(3, \theta, z) = z(5 - z) \sin 3\theta = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} I_n \left( \frac{3m\pi}{5} \right) \sin \left( \frac{m\pi z}{5} \right) [e_{nm} \cos n\theta + f_{nm} \sin n\theta],$$

Since the boundary data is a multiple of $\sin 3\theta$, we know immediately that $e_{nm} = 0$ for all $m$ and $n$ and $f_{nm} = 0$ for all $m$ and all $n$ except $n = 3$. So we can simplify things to finding the coefficients $f_{m3}$ so that

$$z(5 - z) = \sum_{m=1}^{\infty} f_{m3} I_3 \left( \frac{3m\pi}{5} \right) \sin \left( \frac{m\pi z}{5} \right)$$
for \(0 \leq z \leq 5\). This is an ordinary Fourier sine series, so we know that
\[
f_{m3} = \frac{1}{I_3\left(\frac{3m\pi}{5}\right)} \frac{2}{5} \int_0^5 z(5-z) \sin\left(\frac{m\pi z}{5}\right) \, dz.
\]

Work the integral by parts as follows:
\[
\int_0^5 (5z - z^2) \sin\left(\frac{m\pi z}{5}\right) \, dz = -\frac{5}{m\pi} (5z - z^2) \cos\left(\frac{m\pi z}{5}\right)\bigg|_0^5 + \frac{5}{m\pi} \int_0^5 (5-2z) \cos\left(\frac{m\pi z}{5}\right) \, dz
\]
\[
= \frac{25}{m^2\pi^2} (5-2z) \sin\left(\frac{m\pi z}{5}\right)\bigg|_0^5 - \frac{25}{m^2\pi^2} \int_0^5 (-2) \sin\left(\frac{m\pi z}{5}\right) \, dz
\]
\[
= \frac{-250}{m^3\pi^3} \cos\left(\frac{m\pi z}{5}\right)\bigg|_0^5
\]
\[
= \frac{250(1 - (-1)^m)}{m^3\pi^3}
\]
which is zero if \(m\) is even and \(500/(m^3\pi^3)\) if \(m\) is odd. So let \(m = 2k + 1\) and get
\[
f_{(2k+1)3} = \frac{2}{5I_3\left(\frac{3(2k+1)\pi}{5}\right)} \frac{500}{(2k+1)^3\pi^3}.
\]

So the solution of the problem is
\[
u(r, \theta, z) = \sum_{k=0}^\infty \frac{200}{(2k+1)^3\pi^3 I_3\left(\frac{3(2k+1)\pi}{5}\right)} I_3\left(\frac{(2k+1)\pi r}{5}\right) \sin\left(\frac{(2k+1)\pi z}{5}\right) \sin 3\theta
\]

6. Find the first three non-zero terms of the power series (centered at zero) for the solution of the equation
\[
x^2 \frac{d^2 y}{dx^2} - (x^2 + 2)y = 0
\]
that is bounded at \(x = 0\). Does the solution oscillate (like sines and cosines and \(J_n(x)\)) or does it increase steadily to infinity (like \(e^x\) and hyperbolic sines and cosines and \(I_n(x)\))?

The point \(x = 0\) is a regular singular point for this equation, so we must assume that
\[
y = \sum_{n=0}^\infty a_n x^{n+s}
\]
with $a_0 \neq 0$ and discover what values of $s$ work. If we substitute this assumption into the differential equation, we get:

$$\sum_{n=0}^{\infty} (n+s)(n+s-1)a_n x^{n+s} - \sum_{n=0}^{\infty} a_n x^{n+s+2} - \sum_{n=0}^{\infty} 2a_n x^{n+s} = 0.$$ 

Put $m = n + 2$ in the middle term (so $n = m - 2$ and $m$ will go from 2 to $\infty$). Then we can rewrite the whole thing as a single sum from 2 to $\infty$ with the $n=0$ and $n=1$ terms from the first and third sums separated out as follows:

$$[s(s-1)-2]a_0 x^s + [(s+1)s-2]a_1 x^{1+s} + \sum_{n=2}^{\infty} [(((n+s)(n+s-1)) - 2)a_n - a_{n-2}] x^{n+s} = 0$$

In order for $a_0 \neq 0$ we must have $s(s-1) - 2 = s^2 - s - 2 = (s-2)(s+1) = 0$, which gives $s = 2$ or $s = -1$. The solution with $s = -1$ will start with $1/x$ and so cannot be bounded at $x = 0$, so we look only at the $s = 2$ solution, which will start with $x^2$.

Next, the $x^{1+s}$ term has coefficient $[(s+1)s-2]a_1$, which is $4a_1$ for $s = 2$. This tells us that $a_1 = 0$. And from the summation, we get the recurrence relation (using $s = 2$ straightaway)

$$[(n+2)(n+1) - 2]a_n - a_{n-2} = 0$$

or

$$a_n = \frac{a_{n-2}}{n(n+3)}.$$ 

Since $a_1 = 0$, we have $a_n = 0$ for all odd values of $n$. For the first few even values of $n$ we have

$$a_2 = \frac{a_0}{2 \cdot 5}$$

$$a_4 = \frac{a_2}{4 \cdot 7} = \frac{a_0}{(2 \cdot 4)(5 \cdot 7)}$$

$$a_6 = \frac{a_4}{6 \cdot 9} = \frac{a_0}{(2 \cdot 4 \cdot 6)(5 \cdot 7 \cdot 9)}$$

If you think about it long enough, you can see that

$$a_{2n} = \frac{6(n+1)!a_0}{n!(2n+3)!}$$

In any case, the first three non-zero terms are:

$$x^2 + \frac{1}{10} x^4 + \frac{1}{280} x^6 + \ldots$$

This function is always positive and increasing and concave up for $x > 0$ because all the coefficients in its Maclaurin series are positive (and by the ratio test, the series converges for all $x$).