Chapter 1

1. Consider a long thin tube containing a solvent, in which another chemical is dissolved. Let \( u(x,t) \) be the linear density of the chemical (in grams per unit length) \( x \) centimeters from one end of the tube at time \( t \). Suppose more of the chemical is being produced at a rate of \( \alpha u(\beta - u) \) grams per unit length per unit time (we assume the density is constant across each cross-section of the tube, so that \( u \) is only a function of the distance \( x \) along the tube and time \( t \)). Derive the differential equation satisfied by \( u(x,t) \).

2. Suppose the temperature in a rod satisfies

\[
\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} + Lx - \alpha x^2,
\]

and initial condition

\( u(x,0) = 0 \) for \( 0 < x < L \)

and insulated boundary conditions

\[
\frac{\partial u}{\partial x}(0,t) = 0, \quad \frac{\partial u}{\partial x}(L,t) = 0
\]

(a) Find the total energy in the rod at time \( t \). Assume the specific heat of the material in the rod is \( c \), its density is \( \rho \), its thermal conductivity is \( K_0 \) and the cross-sectional area of the rod is \( A \).

(b) For a certain value of \( \alpha \), there is an equilibrium temperature distribution

\[
U(x) = \lim_{t \to \infty} u(x,t),
\]

find \( \alpha \) and the corresponding equilibrium temperature distribution.

Chapter 2

3. Solve the initial/boundary value problem

\[
\begin{align*}
    u_t &= 3u_{xx} - 2u, \\
    u(x,0) &= 5 + 4 \cos 3x, \\
    u_x(0,t) &= 0, \\
    u_x(\pi,t) &= 0.
\end{align*}
\]

What is \( \lim_{t \to \infty} u(x,t) \)?
4. Solve Laplace’s equation

\[ u_{xx} + u_{yy} = 0 \]

on the rectangle \( 0 < x < 10, \ 0 < y < 5 \) with boundary values

\[ u(x, 0) = 0, \quad u(10, y) = y, \quad u(x, 5) = 0, \quad u(0, y) = 5 - y. \]

Chapter 3

5. Let

\[ f(x) = \begin{cases} 
(x - 2)^2 & \text{for } 0 \leq x \leq 2 \\
0 & \text{for } 2 \leq x \leq 4 
\end{cases} \]

(a) Compute the Fourier sine series of \( f(x) \).

(b) Draw a careful graph of the function to which your series converges for \(-12 \leq x \leq 12\).

(c) Compute the Fourier cosine series of \( f(x) \).

(d) Draw a careful graph of the function to which your series converges for \(-12 \leq x \leq 12\).

Chapter 4

6. Suppose a flexible chain of length \( L \) is hanging from the ceiling, and suppose the linear density of the chain is \( \rho \). If we put \( x = 0 \) at the bottom of the chain, and \( x = L \) at the point where the chain is attached to the ceiling, then the magnitude of the tension in the chain at the point \( x \) is the weight of the part of the chain below \( x \), i.e., \( T = \rho gx \) (where \( g \) is the gravitational acceleration). Derive the equation for (small) side-to-side vibrations of the chain.

7. Solve the damped wave equation

\[ u_{tt} = 4u_{xx} - u_t \]

with initial conditions

\[ u(x, 0) = \sin \pi x, \quad u_t(x, 0) = \sin \pi x \]
and boundary conditions
\[ u(0, t) = 0 \quad u(1, t) = 0. \]

Chapter 5

8. (a) Find the eigenvalues and eigenfunctions of the boundary-value problem
\[ x^2 y'' + xy' + \lambda y = 0, \quad y(1) = 0 \quad y(e^\pi) = 0. \]

(b) What definition of the inner product (i.e., what weight function) makes eigenfunctions corresponding to different eigenvalues orthogonal to one another?

Chapter 7

9. (Also chapter 5 and chapter 4)

(a) Use the substitution \( s = 2\sqrt{x} \) to transform the equation
\[ xy'' + y' + \lambda y = 0 \]
into Bessel’s equation.

(b) Find the eigenvalues and eigenfunctions of the problem
\[ xy'' + y' + \lambda y = 0, \quad y(0) \text{ bounded, } y(L) = 0. \]

(c) Solve the initial/boundary value problem
\[ \frac{\partial^2 u}{\partial t^2} = g \left( x \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} \right), \quad u(0, t) \text{ bounded, } u(L, t) = 0 \]
with initial conditions
\[ u(x, 0) = f(x), \quad u_t(x, 0) = 0 \quad \text{for } 0 < x < L. \]

10. (a) Just to prove you can do it, expand the function \( f(r) = r^5 \) on the interval \( 0 < r < 1 \) in a Fourier-Bessel series of the form
\[ r^5 = \sum_{n=1}^{\infty} a_n J_3(z_{3n}r). \]
(b) And now that you’ve done that, solve the heat equation on the disk of radius 1

\[ u_t = 2u_{xx} \]

with zero boundary values and initial values

\[ u(r, \theta, 0) = r^5 \sin 3\theta. \]

11. Find the steady-state temperature \( u(\rho, \varphi, \theta) \) in the solid ball of radius 2 if the surface temperature is given in polar coordinates by

\[ u(2, \varphi, \theta) = \sin^2 \varphi \]

for \( 0 < \varphi < \pi \) in spherical coordinates (here, \( \varphi \) is the elevation angle from the \( xy \)-plane and \( \theta \) is the azimuthal angle).

Chapter 8

12. Solve the inhomogeneous heat equation

\[ \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + r \]

for \( 0 < x < 1, \ t > 0 \) with initial condition

\[ u(x, 0) = 0 \]

and boundary conditions

\[ \frac{\partial u}{\partial x}(0, t) = 0, \quad u(1, t) = 0. \]

13. Solve the wave equation

\[ u_{tt} = u_{xx} \]

for \( 0 < x < \pi \) and \( t > 0 \) with initial data \( u(x, 0) = 0 \) and \( u_t(x, 0) = 0 \) and with boundary data

\[ u(0, t) = 0, \quad u(\pi, t) = \sin t. \]

What happens as \( t \to \infty \)?

Chapter 10
14. Let $f(x) = \begin{cases} 1 & \text{for } 0 < x < 2 \\ 0 & \text{otherwise} \end{cases}$.

(a) Calculate the Fourier transform $\hat{f}(\omega)$.

(b) Solve the initial-value problem for the heat equation

$$u_t = 3u_{xx} \quad u(x, 0) = f(x)$$

for $-\infty < x < \infty$, $t > 0$, with $f(x)$ as given in the beginning of the problem.

15. Consider the initial-value problem for the inhomogeneous heat equation on the whole line:

$$u_t = ku_{xx} + Q(x, t), \quad u(x, 0) = f(x)$$

for $-\infty < x < \infty$, $t > 0$ (and we assume that $Q$ and $f$ and hence $u$ decay at infinity). Let

$$G(x, t) = \frac{1}{\sqrt{4\pi kt}} e^{-x^2/(4kt)}$$

be the fundamental solution of the heat equation (or the “heat kernel”) as defined in the textbook.

Use Fourier transform methods to show that the solution of the problem above is

$$u(x, t) = f \ast G + \int_{-\infty}^{\infty} \int_{0}^{t} Q(y, s) G(x - y, t - s) \, ds \, dy$$

$$= \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} f(y) e^{-(x-y)^2/(4kt)} \, dy + \int_{-\infty}^{\infty} \int_{0}^{t} Q(y, s) \frac{1}{\sqrt{4\pi k(t-s)}} e^{-(x-y)^2/(4k(t-s))} \, ds \, dy.$$  

(Hint: Take the Fourier transform as usual but be very careful how you solve the linear first-order differential equation that results. At some crucial moment you will have to combine some exponentials and change the order of integration.)