Chapter 1

1. Consider a long thin tube containing a solvent, in which another chemical is dissolved. Let $u(x, t)$ be the linear density of the chemical (in grams per unit length) $x$ centimeters from one end of the tube at time $t$. Suppose more of the chemical is being produced at a rate of $\alpha u(\beta - u)$ grams per unit length per unit time (we assume the density is constant across each cross-section of the tube, so that $u$ is only a function of the distance $x$ along the tube and time $t$). Derive the differential equation satisfied by $u(x, t)$.

Let $\varphi(x, t)$ be the flux (from left to right) of the chemical (in grams per unit time) at the point $x$ and time $t$. The rate of change of the amount of chemical between $x = 0$ and $x = b$ is

$$\frac{d}{dt} \int_0^b u(x, t) \, dx = -\varphi(b, t) + \int_0^b \alpha u(\beta - u) \, dx.$$ 

Differentiate with respect to $b$ to get

$$\frac{\partial u}{\partial t} = -\frac{\partial \varphi}{\partial x} + \alpha u(\beta - u).$$

Now use Fick’s law of diffusion which says that the flux is proportional to $-u_x$, so

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} + \alpha u(\beta - u).$$

2. Suppose the temperature in a rod satisfies

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} + Lx - \alpha x^2,$$

and initial condition

$$u(x, 0) = 0 \quad \text{for } 0 < x < L$$

and insulated boundary conditions

$$\frac{\partial u}{\partial x}(0, t) = 0, \quad \frac{\partial u}{\partial x}(L, t) = 0.$$
(a) Find the total energy in the rod at time \( t \). Assume the specific heat of the material in the rod is \( c \), its density is \( \rho \), its thermal conductivity is \( K_0 \) and the cross-sectional area of the rod is \( A \).

(b) For a certain value of \( \alpha \), there is an equilibrium temperature distribution

\[ U(x) = \lim_{t \to \infty} u(x, t), \]

find \( \alpha \) and the corresponding equilibrium temperature distribution.

(a) The total energy in the rod is

\[ E(t) = \int_0^L c\rho A u(x,t) \, dx. \]

Since \( u(x, 0) = 0 \), we have \( E(0) = 0 \). Next,

\[
\frac{dE}{dt} = \int_0^L c\rho A \frac{\partial u}{\partial t} \, dx = c\rho A \left( \int_0^L \left( \frac{k}{c} \frac{\partial^2 u}{\partial x^2} + Lx - \alpha x^2 \right) \, dx \right)
\]

\[
= c\rho A \left( \frac{k}{c} \frac{\partial u}{\partial x} + \frac{Lx^2}{2} - \frac{\alpha x^3}{3} \right) \bigg|_0^L = c\rho A \left( \frac{L^3}{2} - \frac{\alpha L^3}{3} \right)
\]

because of the insulated boundary conditions. Therefore

\[ E(t) = c\rho AL^3 \left( \frac{1}{2} - \frac{\alpha}{3} \right) t \]

(b) There can be no equilibrium unless the total energy stops changing. Since the energy changes linearly with \( t \), this can’t happen unless the energy is constant, so we need \( \alpha = 3/2 \). For this value of \( \alpha \) we will have

\[ kU'' + Lx - \frac{3x^2}{2} = 0, \quad U'(0) = 0, \quad U''(L) = 0 \]

and the total energy in the rod will be zero (since it will not have changed since time \( t = 0 \)), so

\[ \int_0^L U(x) \, dx = 0 \]

From the differential equation we have

\[ U(x) = \frac{1}{k} \left( -\frac{Lx^3}{6} + \frac{x^4}{8} + c_1 x + c_2 \right) \]

so

\[ U'(x) = \frac{1}{k} \left( -\frac{Lx^2}{2} + \frac{x^3}{2} + c_1 \right) \]
This tells us right away that \( c_1 = 0 \) since \( U'(0) = 0 \). It is also clear that \( U'(L) = 0 \) if \( c_1 = 0 \), so we will need the integral condition in order to determine \( c_2 \).

\[
\int_0^L U(x) \, dx = \frac{1}{k} \int_0^L \left( -\frac{Lx^3}{6} + \frac{x^4}{8} + c_2 \right) \, dx = \frac{1}{k} \left( \left. \frac{-Lx^4}{24} + \frac{x^5}{40} + c_2x \right|_0^L \right) = \frac{L}{k} \left( c_2 - \frac{L^4}{60} \right)
\]

so we need \( c_2 = L^4/60 \). Therefore, the equilibrium solution is

\[
U(x) = \frac{1}{k} \left( \frac{L^4}{60} - \frac{Lx^3}{6} + \frac{x^4}{8} \right)
\]

for \( \alpha = 3/2 \).

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**Chapter 2**

3. Solve the initial/boundary value problem

\[
u_t = 3u_{xx} - 2u, \quad u(x, 0) = 5 + 4 \cos 3x, \quad u_x(0, t) = 0, \quad u_x(\pi, t) = 0.
\]

What is \( \lim_{t \to \infty} u(x, t) \) ?

---

Look for solutions of the equation of the form \( u(x, t) = X(x)T(t) \) and find

\[
\frac{T' + 2T}{3T} = \frac{X''}{X} = -\lambda.
\]

Since \( X'(0) = X'() = 0 \), we have \( X(x) = \cos nx \) and \( \lambda = n^2 \), \( n = 0, 1, 2, \ldots \). The corresponding \( T(t) \) satisfies \( T' + 2T = -3n^2T \), so \( T = e^{-(3n^2+2)t} \). So we seek the solution of the problem in the form

\[
u(x, t) = \sum_{n=0}^{\infty} a_n e^{-(3n^2+2)t} \cos nx.
\]

From the initial condition we see that we want \( a_0 = 5 \), \( a_3 = 4 \) and all the other \( a_n = 0 \), and so

\[
u(x, t) = 5e^{-2t} + 4e^{-29t} \cos 3x.
\]

Because both terms have decaying exponential factors, we see that

\[
\lim_{t \to \infty} u(x, t) = 0.
\]

---

4. Solve Laplace’s equation

\[
u_{xx} + u_{yy} = 0
\]
on the rectangle $0 < x < 10$, $0 < y < 5$ with boundary values

$$u(x, 0) = 0, \quad u(10, y) = y, \quad u(x, 5) = 0, \quad u(0, y) = 5 - y.$$ 

We’ll have to do this in two parts, and set $u(x, y) = v(x, y) + w(x, y)$, where $v$ satisfies

$$v(x, 0) = 0, \quad v(10, y) = 0, \quad v(x, 5) = 0, \quad v(0, y) = 5 - y$$

and $w$ satisfies

$$w(x, 0) = 0, \quad w(10, y) = y, \quad w(x, 5) = 0, \quad w(0, y) = 0.$$

We know to look for $v$ in the form

$$v(x, y) = \sum_{n=1}^{\infty} a_n \sinh \left( \frac{n\pi(10 - x)}{5} \right) \sin \left( \frac{n\pi y}{5} \right)$$

and for $w$ in the form

$$w(x, y) = \sum_{n=1}^{\infty} b_n \sinh \left( \frac{n\pi x}{5} \right) \sin \left( \frac{n\pi y}{5} \right)$$

so that each term in $v$ and $w$ satisfies $\nabla^2 v = 0$ and $\nabla^2 w = 0$, and has zero boundary conditions for $y = 0$ and $y = 5$. Next we have to match the non-zero boundary conditions for $v$ and $w$. First,

$$v(0, y) = 5 - y = \sum_{n=1}^{\infty} a_n \sinh (2n\pi) \sin \left( \frac{n\pi y}{5} \right)$$

requires

$$a_n = \frac{2}{5 \sinh (2n\pi)} \int_0^5 (5 - y) \sin \left( \frac{n\pi y}{5} \right) dy$$

$$= \frac{2}{5 \sinh (2n\pi)} \left( -\frac{5 - y}{n\pi} \cos \left( \frac{n\pi y}{5} \right) \right|_0^5 - \frac{5}{n\pi} \int_0^5 \cos \left( \frac{n\pi y}{5} \right) dy \right)$$

$$= \frac{2}{5 \sinh (2n\pi)} \left( \frac{25}{n\pi} - \frac{25}{n^2\pi^2} \sin \left( \frac{n\pi y}{5} \right) \right|_0^5 \right)$$

$$= \frac{10}{n\pi \sinh (2n\pi)}$$

and

$$w(10, y) = y = \sum_{n=1}^{\infty} b_n \sinh (2n\pi) \sin \left( \frac{n\pi y}{5} \right)$$
requires
\[ b_n = \frac{2}{5 \sinh(2n\pi)} \int_0^5 y \sin \left( \frac{n\pi y}{5} \right) \, dy \]
\[ = \frac{2}{5 \sinh(2n\pi)} \left( -y \frac{5}{n\pi} \cos \left( \frac{n\pi y}{5} \right) \bigg|_0^5 + \frac{5}{n\pi} \int_0^5 \cos \left( \frac{n\pi y}{5} \right) \, dy \right) \]
\[ = \frac{2}{5 \sinh(2n\pi)} \left( (-1)^{n+1} \frac{25}{n\pi} - \frac{25}{n^2\pi^2} \sin \left( \frac{n\pi y}{5} \right) \bigg|_0^5 \right) \]
\[ = \frac{(-1)^{n+1}10}{n\pi \sinh(2n\pi)} \]

Therefore,
\[ u(x, y) = \sum_{n=1}^{\infty} \frac{10}{n\pi \sinh(2n\pi)} \left[ \sinh \left( \frac{n\pi(10-x)}{5} \right) + (-1)^{n+1} \sinh \left( \frac{n\pi x}{5} \right) \right] \sin \left( \frac{n\pi x}{5} \right). \]

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Chapter 3

5. Let

\[ f(x) = \begin{cases} 
(x - 2)^2 & \text{for } 0 \leq x \leq 2 \\
0 & \text{for } 2 \leq x \leq 4 
\end{cases} \]

(a) Compute the Fourier sine series of \( f(x) \).

(b) Draw a careful graph of the function to which your series converges for \(-12 \leq x \leq 12\).

(c) Compute the Fourier cosine series of \( f(x) \).

(d) Draw a careful graph of the function to which your series converges for \(-12 \leq x \leq 12\).

(a) The Fourier sine series of \( f(x) \) is
\[ \sum_{n=1}^{\infty} b_n \sin \left( \frac{n\pi x}{4} \right) \]
where

\[
b_n = \frac{1}{2} \int_{0}^{4} f(x) \sin \left( \frac{n\pi x}{4} \right) \, dx = \frac{1}{2} \int_{0}^{2} (x - 2)^2 \sin \left( \frac{n\pi x}{4} \right) \, dx
\]

\[
= -2 \frac{(x - 2)^2}{n\pi} \cos \left( \frac{n\pi x}{4} \right) \bigg|_{0}^{2} + 2 \frac{x}{n\pi} \int_{0}^{2} (x - 2) \cos \left( \frac{n\pi x}{4} \right) \, dx
\]

\[
= 8 \frac{1}{n\pi} + 16 \frac{(x - 2) \sin \left( \frac{n\pi x}{4} \right)}{n^2\pi^2} \bigg|_{0}^{2} - 16 \frac{\sin \left( \frac{n\pi x}{4} \right)}{n^2\pi^2} \bigg|_{0}^{2} \int_{0}^{2} \sin \left( \frac{n\pi x}{4} \right) \, dx
\]

\[
= 8 \frac{1}{n\pi} + \left\{ \begin{array}{ll}
0 & \text{for odd } n \\
\left(-1\right)^{n/2} \frac{64}{n^3\pi^3} & \text{for even } n
\end{array} \right\} - \frac{64}{n^3\pi^3}
\]

(b)

(c) The Fourier cosine series of \( f(x) \) is

\[
a_0 + \sum_{n=1}^{\infty} a_n \cos \left( \frac{n\pi x}{4} \right)
\]

where

\[
a_0 = \frac{1}{4} \int_{0}^{4} f(x) \, dx = \frac{1}{4} \int_{0}^{2} (x - 2)^2 \, dx = \left. \frac{(x - 2)^3}{12} \right|_{0}^{2} = \frac{2}{3}
\]
and for $n \geq 1$

$$a_n = \frac{1}{2} \int_0^4 f(x) \cos \left( \frac{n\pi x}{4} \right) \, dx = \frac{1}{2} \int_0^2 (x-2)^2 \cos \left( \frac{n\pi x}{4} \right) \, dx$$

$$= \frac{2(x-2)^2}{n\pi} \sin \left( \frac{n\pi x}{4} \right) \bigg|_0^2 - \frac{2}{n\pi} \int_0^2 2(x-2) \sin \left( \frac{n\pi x}{4} \right) \, dx$$

$$= \frac{16(x-2)}{n^2\pi^2} \cos \left( \frac{n\pi x}{4} \right) \bigg|_0^2 - \frac{16}{n^2\pi^2} \int_0^2 \cos \left( \frac{n\pi x}{4} \right) \, dx$$

$$= \frac{32}{n^2\pi^2} - \frac{64}{n^3\pi^3} \sin \left( \frac{n\pi x}{4} \right) \bigg|_0^2$$

$$= \frac{32}{n^2\pi^2} + \begin{cases} 
0 & \text{for even } n \\
(-1)^{(n+1)/2} \frac{64}{n^3\pi^3} & \text{for odd } n
\end{cases}$$

(d)

Chapter 4

6. Suppose a flexible chain of length $L$ is hanging from the ceiling, and suppose the linear density of the chain is $\rho$. If we put $x = 0$ at the bottom of the chain, and $x = L$ at the point where the chain is attached to the ceiling, then the magnitude of the tension in the chain at the point $x$ is the weight of the part of the chain below $x$, i.e., $T = \rho g x$ (where $g$ is the gravitational acceleration). Derive the equation for (small) side-to-side vibrations of the chain.

After making the usual approximations (that the points of the chain do not move up and down, and that the angle $\theta$ of deflection of the chain from vertical is small at every point, so we can say that $\theta \approx \tan \theta$), we get that the side-to-side force on the
small segment of the chain between $x$ and $x + \Delta x$ is

$$F = \rho(\Delta x)u_{tt} = T(x + \Delta x) \sin \theta(x + \Delta x) - T(x) \sin \theta(x)$$

$$\approx T(x + \Delta x) \tan \theta(x + \Delta x) - T(x) \tan \theta(x)$$

$$= T(x + \Delta x)u_x(x + \Delta x, t) - T(x)u_x(x, t)$$

Divide both sides by $\rho \Delta x$ and take the limit as $\Delta x \to 0$ and get

$$u_{tt} = \frac{1}{\rho} \frac{\partial}{\partial x} \left( \rho g x \frac{\partial u}{\partial x} \right)$$

or

$$u_{tt} = g(xu_x)_x$$

or

$$u_{tt} = g(xu_{xx} + u_x).$$

7. Solve the damped wave equation

$$u_{tt} = 4u_{xx} - u_t$$

with initial conditions

$$u(x, 0) = \sin \pi x \quad u_t(x, 0) = \sin \pi x$$

and boundary conditions

$$u(0, t) = 0 \quad u(1, t) = 0.$$

Separating variables with $u(x, t) = X(x)T(t)$ gives us

$$\frac{T'' + T'}{4T} = \frac{X''}{X} = -\lambda.$$  

Since $X(0) = X(1) = 0$, we have $X(x) = \sin n\pi x$ and $\lambda = n^2 \pi^2$. This gives us the equation for $T$:

$$T'' + T' + 4n^2 \pi^2 T = 0.$$  

The roots of the auxiliary equation are

$$r_{\pm} = -1 \pm \sqrt{1 - 16n^2 \pi^2}$$

so the solutions for $T$ are

$$T = e^{-t/2} \cos \left( \frac{\sqrt{16n^2 \pi^2 - 1}}{2} t \right) \quad \text{and} \quad e^{-t/2} \sin \left( \frac{\sqrt{16n^2 \pi^2 - 1}}{2} t \right)$$
Because of the initial conditions, we know that we only need the \( n = 1 \) terms, so

\[
    u(x, t) = \left( ae^{-t/2} \cos \left( \frac{\sqrt{16\pi^2 - 1}}{2} t \right) + be^{-t/2} \sin \left( \frac{\sqrt{16\pi^2 - 1}}{2} t \right) \right) \sin \pi x.
\]

We compute that

\[
    u(x, 0) = a \sin \pi x \quad \text{and} \quad u_t(x, 0) = \left( -\frac{a}{2} + \frac{\sqrt{16\pi^2 - 1} b}{2} \right) \sin \pi x.
\]

In order for \( u(x, 0) = \sin \pi x \), we need \( a = 1 \), and then for \( u_t(x, 0) = \sin \pi x \) we need \( b = 3/\sqrt{16\pi^2 - 1} \). Therefore

\[
    u(x, t) = \left( \cos \left( \frac{\sqrt{16\pi^2 - 1}}{2} t \right) + \frac{3}{\sqrt{16\pi^2 - 1}} \sin \left( \frac{\sqrt{16\pi^2 - 1}}{2} t \right) \right) e^{-t/2} \sin \pi x.
\]

**Chapter 5**

8. (a) Find the eigenvalues and eigenfunctions of the boundary-value problem

\[
    x^2 y'' + xy' + \lambda y = 0, \quad y(1) = 0 \quad y(e^\pi) = 0.
\]

(b) What definition of the inner product (i.e., what weight function) makes eigenfunctions corresponding to different eigenvalues orthogonal to one another?

(a) This is a Cauchy-Euler equation, so we guess \( y = x^a \). Substitute this into the equation and get

\[
    (a(a - 1) + a + \lambda) x^a = 0, \quad \text{so} \quad a^2 + \lambda = 0 \quad \text{or} \quad a = \pm \sqrt{-\lambda}.
\]

We have \( y = x^{\pm \sqrt{-\lambda}} \), and we need \( y(1) = 0 \), which can’t happen if \( \lambda \leq 0 \). If \( \lambda > 0 \), then we have

\[
    y = x^{\pm i\sqrt{\lambda}} = e^{\pm i\sqrt{\lambda} \ln x} = \cos(\sqrt{\lambda} \ln x) + i \sin(\sqrt{\lambda} \ln x).
\]

So for \( y(1) = 0 \) we have \( y = \sin(\sqrt{\lambda} \ln x) \). Now we need \( y(e^\pi) = 0 \), and for this we’ll need \( \sqrt{\lambda} \) to be a positive integer. Therefore

\[
    \lambda_n = n^2, \quad n = 1, 2, 3, \ldots
\]

are the eigenvalues and

\[
    y_n(x) = \sin(n \ln x), \quad n = 1, 2, 3, \ldots
\]
are the eigenfunctions for this equation.

(b) To put this equation into Sturm-Liouville form we need to divide by \( x \). This gives:

\[
xy'' + y' + \frac{\lambda}{x} y = \frac{d}{dx} \left( x \frac{dy}{dx} \right) + \frac{\lambda}{x} y = 0.
\]

So the weighting function should be \( 1/x \) and the inner product is

\[
\langle f , g \rangle = \int_1^\pi f(x)g(x) \frac{1}{x} \, dx.
\]

We can check this:

\[
\int_1^\pi \sin(n \ln x) \sin(m \ln x) \, dx = \int_0^\pi \sin(nu) \sin(mu) \, dy = 0
\]

using the \( u \)-substitution \( u = \ln x \) and the fact that \( \sin nu \) and \( \sin mu \) are orthogonal with respect to the weight function \( w(u) = 1 \) on the interval from 0 to \( \pi \).

Chapter 7.

9. (Also chapter 5 and chapter 4)

(a) Use the substitution \( s = 2\sqrt{x} \) to transform the equation

\[
xy'' + y' + \lambda y = 0
\]

into Bessel’s equation.

(b) Find the eigenvalues and eigenfunctions of the problem

\[
xy'' + y' + \lambda y = 0, \quad y(0) \text{ bounded, } y(L) = 0.
\]

(c) Solve the initial/boundary value problem

\[
\frac{\partial^2 u}{\partial t^2} = g \left( x \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} \right), \quad u(0, t) \text{ bounded, } u(L, t) = 0
\]

with initial conditions

\[
u(x, 0) = f(x), \quad u_t(x, 0) = 0 \quad \text{for } 0 < x < L.
\]

(a) Let \( u(s) = y(x(s)) \). Then

\[
\frac{dy}{dx} = \frac{du}{ds} \frac{ds}{dx} = \frac{1}{\sqrt{x}} \frac{du}{ds}
\]
and 
\[ \frac{d^2y}{dx^2} = \frac{d}{dx} \left( \frac{1}{\sqrt{s}} \frac{du}{ds} \right) = \frac{1}{x} \frac{d^2u}{ds^2} - \frac{1}{2x^{3/2}} \frac{du}{ds}. \]

Now we can replace the \( x \)'s with \( s \)'s and get 
\[ \frac{dy}{dx} = \frac{2}{s} \frac{du}{ds} \quad \text{and} \quad \frac{d^2y}{dx^2} = \frac{4}{s^2} \frac{d^2u}{ds^2} - \frac{4}{s^3} \frac{du}{ds}. \]

Put these into the equation 
\[ xy'' + y' + \lambda y = 0 \]

and get 
\[ \frac{s^2}{4} \left( \frac{4}{s^2} \frac{d^2u}{ds^2} - \frac{4}{s^3} \frac{du}{ds} \right) + \frac{2}{s} \frac{du}{ds} + \lambda u = 0, \]

or 
\[ \frac{d^2u}{ds^2} + \frac{1}{s} \frac{du}{ds} + \lambda u = 0. \]

Multiply by \( s^2 \) and get 
\[ s^2 u'' + su' = \lambda s^2 u = 0, \]

which is Bessel’s equation of order zero.

(b) The solutions of Bessel’s equation which are bounded at \( s = 0 \) (since \( x = 0 \) corresponds to \( s = 0 \)) are 
\[ u(s) = cJ_0(\sqrt{\lambda} s). \]

This makes the solution of the original equation \( xy'' + y' + \lambda y = 0 \)
\[ y(x) = cJ_0(2\sqrt{\lambda} x). \]

For \( y(L) = 0 \), we need \( 2\sqrt{\lambda L} \) to be \( z_0 \) (the \( n \)th positive zero of the Bessel function \( J_0 \)). Therefore
\[ \lambda = \frac{z_0^2}{4L} \]

are the eigenvalues and 
\[ J_0 \left( \frac{z_0}{\sqrt{L}} \right) \]

are the eigenfunctions of this operator.

(c) We separate variables in the problem to get 
\[ \frac{T''}{gT} = \frac{xX'' + X'}{X} = -\lambda. \]

The \( X \) problem is the one we just solved, namely 
\[ xX'' + X' + \lambda X = 0, \quad X(0) \text{ bounded,} \quad X(L) = 0, \]
so we have

$$\lambda = \frac{z_0^2}{4L} \quad \text{and} \quad X(x) = J_0 \left( z_0 \sqrt{\frac{x}{L}} \right)$$

and The $T$ solutions are thus

$$T = a_n \cos \left( \sqrt{\frac{g}{L}} \frac{z_0 t}{2} \right) + b_n \sin \left( \sqrt{\frac{g}{L}} \frac{z_0 t}{2} \right).$$

We’ll only need the cosine terms since $u_t(x,0) = 0$. So our solution will have the form

$$u(x, t) = \sum_{n=1}^{\infty} a_n J_0 \left( z_0 \sqrt{\frac{x}{L}} \right) \cos \left( \sqrt{\frac{g}{L}} \frac{z_0 t}{2} \right)$$

and we are left with the task of determining the constants $a_n$.

Now our problem was already in Sturm-Liouville form, and so the weight function $w(x) = 1$. So for

$$u(x, 0) = f(x) = \sum_{n=1}^{\infty} a_n J_0 \left( z_0 \sqrt{\frac{x}{L}} \right)$$

we will need

$$a_n = \frac{\langle f(x) , J_0 \left( z_0 \sqrt{\frac{x}{L}} \right) \rangle}{\langle J_0 \left( z_0 \sqrt{\frac{x}{L}} \right) , J_0 \left( z_0 \sqrt{\frac{x}{L}} \right) \rangle} = \frac{\int_0^L f(x) J_0 \left( z_0 \sqrt{\frac{x}{L}} \right) dx}{\int_0^L \left( J_0 \left( z_0 \sqrt{\frac{x}{L}} \right) \right)^2 dx}$$

You can make the $s = 2\sqrt{x}$ substitution in the integral in the denominator to evaluate it — it’s $(J_1(z_0))^2 L$. But I’ll leave that to you. Note that this is the solution of the hanging chain problem (problem 6 above).

10. (a) Just to prove you can do it, expand the function $f(r) = r^5$ on the interval $0 < r < 1$ in a Fourier-Bessel series of the form

$$r^5 = \sum_{n=1}^{\infty} a_n J_3(z_3 r).$$

(b) And now that you’ve done that, solve the heat equation on the disk of radius 1

$$u_t = 2u_{xx}$$

with zero boundary values and initial values

$$u(r, \theta, 0) = r^5 \sin 3\theta.$$
(a) In the sum, we know that

\[
 a_n = \frac{\langle r^5, J_3(z_{3n}r) \rangle}{\langle J_3(z_{3n}r), J_3(z_{3n}r) \rangle} = \frac{\int_0^1 r^5 J_3(z_{3n}r) \, r \, dr}{\int_0^1 (J_3(z_{3n}r))^2 \, r \, dr} = \frac{2}{(J_4(z_{3n}))^2} \int_0^1 r^5 J_3(z_{3n}r) \, r \, dr.
\]

For the moment, let’s simply write \( z \) for \( z_{3n} \) so we don’t have to keep copying the subscripts over and over. Now we’ll start by making the substitution \( x = zr \) (so \( r = x/z \) and \( dr = dx/z \)), then we’ll integrate by parts with \( u = x^2 \) etc:

\[
\int_0^1 r^5 J_3(z_{3n}r) \, r \, dr = \frac{1}{z^7} \int_0^z x^6 J_3(x) \, dx \\
= \frac{1}{z^7} \left( x^6 J_4(x) \bigg|_0^z - \int_0^z 2x^5 J_4(x) \, dx \right) \\
= \frac{1}{z^7} (z^6 J_4(z) - 2z^5 J_5(z)) = \frac{z J_4(z) - 2 J_5(z)}{z^2}
\]

We can now use the identity

\[
 J_{n+1}(x) = \frac{2n}{x} J_n(x) - J_{n-1}(x)
\]

with \( n = 4 \) (keeping in mind that \( J_3(z) = 0 \)) and then with \( n = 3 \) to replace \( J_5(z) \) with

\[
 J_5(z) = \frac{8}{z} J_4(z) - J_3(z) = \frac{8}{z} J_4(z) = \frac{8}{z} \left( \frac{6}{z} J_3(z) - J_2(z) \right) = -\frac{8}{z} J_2(z),
\]

and note that \( J_4(z) = -J_2(z) \), which simplifies our integral to

\[
\int_0^1 r^5 J_3(z_{3n}r) \, r \, dr = \frac{16 - z^2}{z^3} J_2(z).
\]

Therefore

\[
 a_n = \frac{32 - 2z_{3n}^2}{z_{3n}^3 J_2(z_{3n})}.
\]

(We could use the identity again to replace \( J_2 \) with \( J_1 \) and \( J_0 \) but that seems to me to make it more complicated.) Thus

\[
r^5 = \sum_{n=1}^{\infty} \frac{32 - 2z_{3n}^2}{z_{3n}^3 J_2(z_{3n})} J_3(z_{3n}r).
\]
(b)  In polar coordinates,\[ u_t = 2 \left( \frac{1}{r} (ru_r)_r + \frac{1}{r^2} u_{\theta\theta} \right) \]

Separating variables \( u(r, \theta, t) = R(r) \Theta(\theta) T(t) \) gives\[ \frac{T'}{2T} = \frac{(rR')'}{rR} + \frac{\Theta''}{r^2 \Theta} = -\lambda \]

so\[ T = ce^{-2\lambda t}. \]

Next, multiply the right separated equation by \( r^2 \) and rearrange to get\[ \frac{r^2 R'' + rR'}{R} + \lambda r^2 = -\frac{\Theta''}{\Theta} = \mu \]

and since \( \Theta \) must be periodic, we have that \( \mu = m^2 \) and \( \Theta = a_m \cos m\theta + b_m \sin m\theta. \)

The \( R \) equation becomes\[ r^2 R'' + rR' + (\lambda r^2 - m^2) R = 0 \]

which is Bessel’s equation of order \( m \). Since we need \( R(1) = 0 \) from the boundary condition (and \( R(0) \) is bounded) we have\[ R(r) = J_m(z_{mn} r) \quad \text{and} \quad \lambda = z_{mn}^2 \]

(or \( R = 1 \) for \( \lambda = 0 \)). Thus, the solution is\[ u(r, \theta, t) = a_{00} + \sum_{n=1}^{\infty} a_{0n} e^{-2z_{0n}^2 t} J_0(z_{0n} r) + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} e^{-2z_{mn}^2 t} J_m(z_{mn} r) [a_{mn} \cos m\theta + b_{mn} \sin m\theta]. \]

But the initial condition \( u(r, \theta, 0) = r^5 \sin 3\theta \) tells us that all the \( a_{mn} \)'s are zero and all the \( a_{mn} \)'s are zero except when \( n = 3 \). So the solution reduces to\[ u(r, \theta, t) = \sum_{n=1}^{\infty} a_{3n} e^{-2z_{3n}^2 t} J_3(z_{3n} r) \sin 3\theta. \]

Applying the initial condition and the solution to part (a) gives the final answer:

\[ u(r, \theta, t) = \sum_{n=1}^{\infty} \frac{32 - 2z_{3n}^2}{z_{3n}^3 J_2(z_{3n})} e^{-2z_{3n}^2 t} J_3(z_{3n} r) \sin 3\theta. \]
11. Find the steady-state temperature \( u(\rho, \varphi, \theta) \) in the solid ball of radius 2 if the surface temperature is given in polar coordinates by

\[
u(2, \varphi, \theta) = \sin^2 \varphi
\]

for \( 0 < \varphi < \pi \) in spherical coordinates (here, \( \varphi \) is the elevation angle from the \( xy \)-plane and \( \theta \) is the azimuthal angle).

We need to find the solution of Laplace’s equation \( \nabla^2 u = 0 \) with the given boundary values. In spherical coordinates,

\[
\nabla^2 u = \frac{\partial^2 u}{\partial \rho^2} + \frac{2}{\rho} \frac{\partial u}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 u}{\partial \varphi^2} + \cot \varphi \frac{\partial u}{\partial \varphi} + \frac{1}{\rho^2 \sin^2 \varphi} \frac{\partial^2 u}{\partial \theta^2}.
\]

Since the boundary data is independent of \( \theta \), the solution will be as well, so the Laplacian reduces to

\[
\nabla^2 u = \frac{\partial^2 u}{\partial \rho^2} + \frac{2}{\rho} \frac{\partial u}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 u}{\partial \varphi^2} + \cot \varphi \frac{\partial u}{\partial \varphi}.
\]

We separate variables as \( u(\rho, \varphi) = R(\rho) \Phi(\varphi) \) and get

\[
\frac{\rho^2 R'' + 2 \rho R'}{R} = -\frac{\Phi'' + \cot \varphi \Phi'}{\Phi} = \lambda
\]

so we have

\[
\rho^2 R'' + 2 \rho R' - \lambda R = 0 \quad \text{and} \quad \sin \varphi \Phi'' + \cos \varphi \Phi' + \lambda \sin \varphi \Phi = 0
\]

where we need \( \Phi(\varphi) \) to be bounded when \( \varphi = 0 \) and \( \varphi = \pi \). Make the substitution \( x = \cos \varphi \) in the second equation and it becomes

\[
(1 - x^2) \frac{d^2 \Phi}{dx^2} - 2x \frac{d \Phi}{dx} + \lambda \Phi = 0
\]

for \(-1 < x < 1\), which is Legendre’s equation. The only solutions of this which are bounded for \( x = -1 \) and \( x = 1 \) are the Legendre polynomials \( P_n(x) \) which means that \( \lambda = n(n + 1) \) for \( n = 0, 1, 2, \ldots \). So the eigenvalues and eigenfunctions of the second separated equation are

\[
\lambda = n(n + 1), \quad \Phi(\varphi) = P_n(\cos \varphi).
\]

Next, we consider the solutions of

\[
\rho^2 R'' + 2 \rho R' - (n + 1)R = 0
\]
which is a Cauchy-Euler equation. As usual, we guess $R = \rho^a$ and see that $a(a-1) + 2a - n(n+1) = a(a+1) - n(n+1) = 0$ so $a = n$ or $a = -(n+1)$. To have $R(\rho)$ bounded at $\rho = 0$ we use only $a = n$ and so we conclude that

$$u(\rho, \varphi) = \sum_{n=1}^{\infty} a_n \rho^n P_n(\cos \varphi).$$

When $\rho = 2$ we need

$$\sin^2 \varphi = \sum_{n=1}^{\infty} 2^n a_n P_n(\cos \varphi).$$

Now $P_0(\cos \varphi) = 1$, $P_1(\cos \varphi) = \cos \varphi$ and $P_2(\cos \varphi) = \frac{1}{2}(3 \cos^2 \varphi - 1)$. Thus

$$\sin^2 \varphi = 1 - \cos^2 \varphi = \frac{2}{3} P_0(\cos \varphi) - \frac{2}{3} P_2(\cos \varphi)$$

so we need $a_0 = \frac{2}{3}$, $a_2 = -\frac{1}{6}$ and all the other $a_n = 0$. We conclude that

$$u(\rho, \varphi) = \frac{2}{3} P_0(\cos \varphi) - \frac{1}{6} \rho^2 P_2(\cos \varphi) = \frac{2}{3} - \frac{1}{12} \rho^2 (3 \cos^2 \varphi - 1) = \frac{2}{3} + \frac{1}{12} \rho^2 - \frac{1}{4} \rho^2 \cos^2 \varphi$$

(Note, in rectangular coordinates, $u = \frac{2}{3} + \frac{1}{12} (x^2 + y^2 - 2z^2)$).

Chapter 8

12. Solve the inhomogeneous heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + r$$

for $0 < x < 1$, $t > 0$ with initial condition

$$u(x, 0) = 0$$

and boundary conditions

$$\frac{\partial u}{\partial x}(0, t) = 0, \quad u(1, t) = 0.$$

The eigenfunctions of the problem

$$X'' + \lambda X = 0, \quad X'(0) = 0, \quad X(1) = 0$$

are

$$X = \cos \left( \frac{(2n+1)\pi x}{2} \right), \quad \text{so} \quad \lambda = \frac{(2n+1)^2 \pi^2}{4}$$
for \( n = 0, 1, 2, \ldots \). So we seek the solution of the problem in the form

\[
u(x, t) = \sum_{n=0}^{\infty} a_n(t) \cos \left( \frac{(2n + 1)\pi x}{2} \right).
\]

Substitute this into the equation and get

\[
\sum_{n=0}^{\infty} \left( a_n'(t) + \frac{(2n + 1)^2 \pi^2}{4} a_n(t) \right) \cos \left( \frac{(2n + 1)\pi x}{2} \right) = r.
\]

So we need the Fourier series for the constant function \( r \):

\[
r = \sum_{n=0}^{\infty} b_n \cos \left( \frac{(2n + 1)\pi x}{2} \right)
\]

where

\[
b_n = 2 \int_0^1 r \cos \left( \frac{(2n + 1)\pi x}{2} \right) \, dx = \frac{4r}{(2n + 1)\pi} \sin \left( \frac{(2n + 1)\pi x}{2} \right) \bigg|_0^1 = (-1)^n \frac{4r}{(2n + 1)\pi}.
\]

Therefore we need \( a_n(t) \) to satisfy

\[
a_n' + \frac{(2n + 1)^2 \pi^2}{4} a_n = (-1)^n \frac{4r}{(2n + 1)\pi}, \quad a_n(0) = 0.
\]

So (using the method of undetermined coefficients, or solving this as a linear equation)

\[
a_n = (-1)^n \frac{16r}{(2n + 1)^3 \pi^3} \left( 1 - e^{-(2n + 1)^2 \pi^2 t/4} \right).
\]

Therefore

\[
u(x, t) = \sum_{n=0}^{\infty} (-1)^n \frac{16r}{(2n + 1)^3 \pi^3} \left( 1 - e^{-(2n + 1)^2 \pi^2 t/4} \right) \cos \left( \frac{(2n + 1)\pi x}{2} \right).
\]

13. Solve the wave equation

\[
u_{tt} = \nu_{xx}
\]

for \( 0 < x < \pi \) and \( t > 0 \) with initial data \( \nu(x, 0) = 0 \) and \( \nu_t(x, 0) = 0 \) and with boundary data

\[
u(0, t) = 0, \quad \nu(\pi, t) = \sin t.
\]

What happens as \( t \to \infty \)?
We first need to get rid of the inhomogeneous boundary condition, so set
\[ u(x, t) = v(x, t) + \frac{x}{\pi} \sin t. \]
Then \( u_{xx} = v_{xx} \) but \( u_{tt} = v_{tt} - \frac{x}{\pi} \sin t \), so \( v \) satisfies the equation
\[ v_{tt} = v_{xx} + \frac{x}{\pi} \sin t. \]
The boundary conditions for \( v \) are homogeneous: \( v(0, t) = v(\pi, t) = 0 \), and the initial position for \( v \) is homogeneous as well: \( v(x, 0) = 0 \). But
\[ u_t = v_t + \frac{x}{\pi} \cos t, \]
so
\[ v_t(x, 0) = -\frac{x}{\pi}. \]
We’re definitely going to need the Fourier sine series of \( x \), which is
\[ x = \sum_{n=1}^{\infty} b_n \sin nx, \]
where
\[ b_n = \frac{2}{\pi} \int_0^\pi x \sin nx \, dx = \frac{2}{\pi} \left( -\frac{x}{n} \cos nx \right|_0^\pi + \frac{1}{n} \int_0^\pi \cos nx \, dx \right) \]
\[ = \frac{2}{\pi} \left( \frac{(-1)^{n+1} \pi}{n} + \frac{1}{n^2} \sin nx \right|_0^\pi \right) \]
\[ = \frac{(-1)^{n+1} 2}{n}. \]
Next, we set
\[ v(x, t) = \sum_{n=1}^{\infty} a_n(t) \sin nx. \]
From the differential equation, we have
\[ \sum_{n=1}^{\infty} \left( a''_n + n^2 a_n \right) \sin nx = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 2}{n} \sin t \sin nx \]
and the initial conditions are given by
\[ v(x, 0) = \sum_{n=1}^{\infty} a_n(0) \sin nx = 0 \quad \text{and} \quad v_t(x, 0) = \sum_{n=1}^{\infty} a'_n(0) \sin nx = \sum_{n=1}^{\infty} \frac{(-1)^n 2}{n\pi} \sin nx = -\frac{x}{\pi}. \]
So for each \( n = 1, 2, 3, \ldots \) we have
\[ a''_n + n^2 a_n = \frac{(-1)^{n+1} 2}{n\pi} \sin t, \quad a_n(0) = 0 \quad a'_n(0) = \frac{(-1)^n 2}{n\pi}. \]
By the method of undetermined coefficients, for \( n > 1 \) we guess a particular solution of the form \( A \sin x \), and the general solution of the differential equation is

\[
a_n(t) = \frac{(-1)^{n+1}2}{n(n^2 - 1)\pi} \sin t + c_n \cos nt + d_n \sin nt.
\]

Since \( a_n(0) = 0 \), we get that \( c_n = 0 \). For \( a_n'(0) \) we have to choose \( d_n \) so that

\[
n d_n + \frac{(-1)^{n+1}2}{n(n^2 - 1)\pi} = \frac{(-1)^n2}{n\pi}
\]

so (after some algebra)

\[
d_n = \frac{(-1)^n2}{(n^2 - 1)\pi}.
\]

We conclude that for \( n > 1 \) we have

\[
a_n(t) = \frac{(-1)^{n+1}2}{n(n^2 - 1)\pi} \sin t + \frac{(-1)^n2}{(n^2 - 1)\pi} \sin nt.
\]

For \( n = 1 \) we have to guess that the particular solution is of the form \( At \cos t \). The second derivative of \( At \cos t \) is \( -2A \sin t - At \cos t \), so the differential equation becomes

\[
-2A \sin t = \frac{2}{\pi} \sin t
\]

and we conclude that \( A = -1/\pi \), so our general solution for \( n = 1 \) is of the form

\[
a_1(t) = -\frac{1}{\pi} t \cos t + c_1 \cos t + d_1 \sin t.
\]

Again, since \( a_1(0) = 0 \), we have \( c_1 = 0 \). For \( a_1'(0) \) we have to choose \( d_1 \) so that

\[
d_1 - \frac{1}{\pi} = -\frac{2}{\pi},
\]

and we conclude that \( d_1 = -1/\pi \), so that

\[
a_1(t) = -\frac{1}{\pi} t \cos t - \frac{1}{\pi} \sin t.
\]

Altogether then, we have

\[
v(x, t) = \left( -\frac{1}{\pi} t \cos t - \frac{1}{\pi} \sin t \right) \sin x + \sum_{n=2}^{\infty} \left( \frac{(-1)^{n+1}2}{n(n^2 - 1)\pi} \sin t + \frac{(-1)^n2}{(n^2 - 1)\pi} \sin nt \right) \sin nx
\]

\[
= -\frac{1}{\pi} (t \cos t + \sin t) \sin x + \sum_{n=2}^{\infty} \frac{(1)^{n+1}2}{n(n^2 - 1)\pi} \left( \frac{1}{n} \sin t - \sin nt \right) \sin nx.
\]

And finally,

\[
u(x, t) = \frac{x}{\pi} \sin t - \frac{1}{\pi} (t \cos t + \sin t) \sin x + \sum_{n=2}^{\infty} \frac{(1)^{n+1}2}{n(n^2 - 1)\pi} \left( \frac{1}{n} \sin t - \sin nt \right) \sin nx.
\]
Because of the $t \cos t$ term, the basic sine wave gets amplified and the amplitude of the vibrations of the string increases without bound — this is because of resonance between the frequency of the input at the $x = \pi$ endpoint and the first natural frequency (normal mode) of vibration of the string.

Chapter 10

14. Let $f(x) = \begin{cases} 1 & \text{for } 0 < x < 2 \\ 0 & \text{otherwise} \end{cases}$.

(a) Calculate the Fourier transform $\hat{f}(\omega)$.

(b) Solve the initial-value problem for the heat equation

$$u_t = 3u_{xx}, \quad u(x, 0) = f(x)$$

for $-\infty < x < \infty$, $t > 0$, with $f(x)$ as given in the beginning of the problem.

(a) We have

$$\hat{f}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x)e^{i\omega x} dx = \frac{1}{2\pi} \int_{0}^{2} e^{i\omega x} dx = \frac{e^{i\omega x}}{2\pi i\omega} \bigg|_{0}^{2} = \frac{e^{2i\omega} - 1}{2\pi i\omega}$$

If we factor out $e^{i\omega}$ from the numerator, we can rewrite this as

$$\hat{f}(\omega) = e^{i\omega} \frac{e^{i\omega} - e^{-i\omega}}{2\pi i\omega} = e^{i\omega} \sin \frac{\omega}{\pi\omega}.$$ 

Note that we could have done this using the known formula

$$\mathcal{F} [S_a(x)] = \frac{\sin a\omega}{\pi\omega}$$

together with the shift formula

$$\mathcal{F} [f(x - a)] = e^{ia\omega} \hat{f}(\omega),$$

since $f(x) = S_1(x - 1)$. So we would immediately get

$$\hat{f}(\omega) = \mathcal{F} [S_1(x - 1)] = e^{i\omega} \sin \frac{\omega}{\pi\omega}.$$ 

(b) We take the Fourier transform in $x$ of everything and get

$$\hat{u}_t + 3\omega^2 \hat{u} = 0, \quad \hat{u}(\omega, 0) = e^{i\omega} \sin \frac{\omega}{\pi\omega}.$$
Therefore
\[ \hat{u}(\omega, t) = \frac{e^{i\omega \tau} e^{-3\omega^2 t} \sin \omega}{\pi \omega}. \]

To recover \( u \), we can use the shifting rule (that will take care of the \( e^{i\omega} \), the convolution rule to handle the product of the other two factors, and the two rules we know, to obtain
\[ u(x, t) = \frac{1}{2\pi} S_1(x - 1) \ast \mathcal{F}^{-1} \left[ e^{-3\omega^2 t} \right] \]
The rule for Gaussians is
\[ \mathcal{F} \left[ e^{-a x^2/2} \right] = \frac{1}{\sqrt{2\pi a}} e^{-\omega^2/(2a)}, \]
so we need \( 3t = 1/(2a) \), or \( a = 1/(6t) \), and then
\[ \mathcal{F}^{-1} \left[ e^{-3\omega^2 t} \right] = \sqrt{\frac{\pi}{3t}} e^{-x^2/(12t)}. \]

Altogether, we get
\[ u(x, t) = \frac{1}{\sqrt{12\pi t}} \int_{-\infty}^{\infty} S_1(y - 1) e^{-(x-y)^2/(12t)} dy = \frac{1}{\sqrt{12\pi t}} \int_0^2 e^{-(x-y)^2/(12t)} dy. \]

Perhaps this is going a little overboard, but we can express this in terms of the error function:
\[ \text{erf}(x) = \frac{2}{\pi} \int_0^x e^{-u^2} du \]
To do this, make the substitution \( u = (x - y)/\sqrt{12t} \), so \( du = -dy/\sqrt{12t} \). Then
\[ u(x, t) = \frac{1}{\sqrt{12\pi t}} \int_0^2 e^{-(x-y)^2/(12t)} dy = -\frac{1}{\sqrt{\pi}} \int_{x/\sqrt{12t}} \sqrt{\frac{\pi}{2}} \left[ \text{erf} \left( \frac{x}{\sqrt{12t}} \right) - \text{erf} \left( \frac{x - 2}{\sqrt{12t}} \right) \right]. \]

15. Consider the initial-value problem for the inhomogeneous heat equation on the whole line:
\[ u_t = k u_{xx} + Q(x, t), \quad u(x, 0) = f(x) \]
for \(-\infty < x < \infty, t > 0\) (and we assume that \( Q \) and \( f \) and hence \( u \) decay at infinity). Let
\[ G(x, t) = \frac{1}{\sqrt{4\pi kt}} e^{-x^2/(4kt)}. \]
be the fundamental solution of the heat equation (or the “heat kernel”) as defined in the textbook.

Use Fourier transform methods to show that the solution of the problem above is

$$u(x, t) = f \ast G + \int_{-\infty}^{\infty} \int_{0}^{t} Q(y, s) G(x - y, t - s) \, ds \, dy$$

$$= \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} f(y) e^{-(x-y)^2/(4kt)} \, dy + \int_{-\infty}^{\infty} \int_{0}^{t} Q(y, s) \frac{1}{\sqrt{4\pi k(t - s)}} e^{-(x-y)^2/(4k(t-s))} \, ds \, dy.$$  

(Hint: Take the Fourier transform as usual but be very careful how you solve the linear first-order differential equation that results. At some crucial moment you will have to combine some exponentials and change the order of integration.)

We take the Fourier transform of the whole problem and get

$$\hat{u}_t = -k\omega^2 \hat{u} + \hat{Q}(\omega, t), \quad \hat{u}(\omega, 0) = \hat{f}(\omega).$$

The first-order differential equation for $\hat{u}$ (considered as a function of $t$, with $\omega$ just going along for the ride) is

$$\hat{u}' + k\omega^2 \hat{u} = \hat{Q}(\omega, t).$$

The integrating factor for this is $e^{k\omega^2 t}$, so we have

$$\left(e^{k\omega^2 t} \hat{u}(\omega, t)\right)' = e^{k\omega^2 t} \hat{Q}(\omega, t).$$

Change $t$ to another letter (variable of integration), say $s$, and integrate both sides from 0 to $t$ and get

$$\int_{0}^{t} \frac{d}{ds} \left(e^{k\omega^2 s} \hat{u}(\omega, s)\right) \, ds = \int_{0}^{t} e^{k\omega^2 s} \hat{Q}(\omega, s) \, ds$$

$$e^{k\omega^2 t} \hat{u}(\omega, t) - \hat{u}(\omega, 0) = \int_{0}^{t} e^{k\omega^2 s} \hat{Q}(\omega, s) \, ds$$

$$e^{k\omega^2 t} \hat{u}(\omega, t) - \hat{f}(\omega) = \int_{0}^{t} e^{k\omega^2 s} \hat{Q}(\omega, s) \, ds.$$  

Therefore

$$\hat{u}(\omega, t) = e^{-k\omega^2 t} \hat{f}(\omega) + e^{-k\omega^2 t} \int_{0}^{t} e^{k\omega^2 s} \hat{Q}(\omega, s) \, ds$$

$$= e^{-k\omega^2 t} \hat{f}(\omega) + \int_{0}^{t} e^{-k\omega^2 (t-s)} \hat{Q}(\omega, s) \, ds.$$
Now we take the inverse Fourier transform. The first term is the usual convolution of the heat kernel with the initial data that we get for the homogeneous problem. In the second term, we can bring the inverse Fourier transform integral inside the $ds$ integral and recognize that we get the convolution of the heat kernel (evaluated at time $t - s$) with the function $Q$ (evaluated at time $s$). In other words:

$$u(x, t) = G(x, t) * f(x) + \int_0^t G(x, t - s) * Q(x, s) \, ds$$

$$= \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} f(y) e^{-(x-y)^2/(4kt)} \, dy + \int_0^t \int_{-\infty}^{\infty} Q(y, s) \frac{1}{\sqrt{4\pi k(t-s)}} e^{-(x-y)^2/(4k(t-s))} \, dy \, ds$$

$$= \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} f(y) e^{-(x-y)^2/(4kt)} \, dy + \int_{-\infty}^{t} \int_0^t Q(y, s) \frac{1}{\sqrt{4\pi k(t-s)}} e^{-(x-y)^2/(4k(t-s))} \, ds \, dy$$

upon reversing the order of integration. Note that if we define $Q(x, t)$ and $G(x, t)$ to be zero for $t < 0$, then the second term is the “double convolution” of $Q$ with $G$. 