1. Let $V$ and $W$ be vector spaces, and let $T: V \rightarrow W$ be a linear map. Suppose $Tv = a$, $Tw = b$ (with $a \neq b$) and $Tx = a + b$. Must $x = v + w$? Give a proof or a counterexample (in which you get to pick the spaces, the map $T$ and the vectors, as long as they satisfy the given conditions).

The answer to the question is no. If $\dim \ker T \geq 1$, then there are non-zero vectors $y$ that satisfy $Ty = 0$, and we could have $x = v + w + y$. So for example if $V = \mathbb{R}^3$ and $W = \mathbb{R}^2$ and the matrix of $T$ is

$$
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0
\end{bmatrix}
$$

we have

$$
T \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad T \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad T \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}.
$$

So

$$
x = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = v + w.
$$

2. (a) Let $V$ be a four-dimensional vector space, and suppose $T: V \rightarrow V$. Let $v_1$, $v_2$ and $v_3$ be linearly independent vectors such that $Tv_1 = 0$, $Tv_2 = w \neq 0$ and $Tv_3 = 2w$. What can you say about the dimension of the image of $T$?

(b) Let $S: \mathbb{R}^2 \rightarrow \mathbb{R}^5$ and $T: \mathbb{R}^8 \rightarrow \mathbb{R}^5$ be linear maps. Suppose $\text{im}(T) \subset \text{im}(S)$. What can you say about the dimension of $\ker(T)$?

(a) We know that $v_1$ and $v_3 - 2v_2$ are linearly dependent vectors in $\ker T$, therefore $\dim \ker T \geq 2$. Since we know that $\dim \ker T + \dim \text{im} T = 4$, it must also be true that $\dim \text{im} T \leq 2$.

(b) Since the image of $T$ is contained in the image of $S$, we have $\dim \text{im} T \leq \dim \text{im} S \leq 2$. And since $\dim \ker T + \dim \text{im} T = 8$, we must have $\dim \ker T \geq 6$.

3. Given a system of 4 linear equations in 6 variables, one begins with the augmented matrix and obtains the following row-echelon form:

$$
\begin{bmatrix}
1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & a
\end{bmatrix}
$$

(a) For what value(s) of $a$ does this system have solutions?

(b) What is the dimension of the space of solutions of the related homogeneous equations?

(c) Find a basis of the space of solutions of the related homogeneous system.
(d) Find a particular solution of the original system for each value of $a$ you found in part (a).

(e) Express all the solutions of the original system for each value of $a$ you found in part (a).

(a) Since the last equation says $0 = a$, the only value of $a$ for which the system can have solutions is $a = 0$.

(b) There are three independent equations in six variables (and there are three columns in which no rows “start”), so the dimension of the homogeneous solution space is 3.

(c) A basis consists of the vectors

$$\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}.$$ 

(d) For $a = 0$, a particular solution is

$$\begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

(e) The general solution is

$$\begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}.$$

4. Let $\mathcal{P}_3$ be the space of polynomials of degree 3 or less, and let

$$\{p_0(x) = 1, p_1(x) = x, p_2(x) = x^2, p_3(x) = x^3\}$$

be the standard basis for $\mathcal{P}_3$.

(a) Find the matrix of the linear map $L: \mathcal{P}_3 \rightarrow \mathcal{P}_3$ defined by

$$L(p) = p'' + 2p' + 2p$$

with respect to the standard basis.

(b) Find the eigenvalues and eigenvectors (eigenfunctions) of $L$. Is $L$ diagonalizable?

(c) What is the determinant of $L$? If $L$ is invertible, find its inverse.

(a) Since $L(p_0) = L(1) = 2 = 2p_0$, $L(p_1) = L(x) = 2 + 2x = 2p_0 + 2p_1$, $L(p_2) = L(x^2) = 2 + 4x + 2x^2 = 2p_0 + 4p + 1 + 2p_2$ and $L(p_3) = L(x^3) = 6x + 6x^2 + 2x^3 = 6p_1 + 6p_2 + 2p_3$, the matrix
of $L$ is
\[ L = \begin{bmatrix} 2 & 2 & 2 & 0 \\ 0 & 2 & 4 & 6 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 2 \end{bmatrix}. \]

(b) $L$ is upper triangular, so the eigenvalues of $L$ are the diagonal entries, i.e., $\lambda = 2$ is the only eigenvalue. To find eigenvectors, row reduce
\[ \begin{bmatrix} 0 & 2 & 2 & 0 \\ 0 & 0 & 4 & 6 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \]
so the only eigenvector of $L$ is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, i.e., the eigenfunctions are constants. Since $L$ has only one linearly independent eigenvector, it is not diagonalizable.

(c) The determinant of $L$ is the product of its diagonal entries, so $\det L = 16$, and so $L$ is invertible. To find $L^{-1}$, row reduce (working from the bottom up):
\[ \begin{bmatrix} 2 & 2 & 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 4 & 6 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 6 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 2 & 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 4 & 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 2 & 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1/2 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 2 & 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 4 & 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 2 & 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1/2 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 0 & 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1/2 & -1 & 3/2 \\ 0 & 0 & 1 & 0 & 1/2 & -3/2 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1/2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 1/2 & -3/2 \\ 0 & 1 & 0 & 0 & 1/2 & -3/2 \\ 0 & 0 & 1 & 0 & 1/2 & -3/2 \\ 0 & 0 & 0 & 1 & 1/2 & 0 \end{bmatrix} 
\]
Therefore
\[ L^{-1} = \begin{bmatrix} 1/2 & -1/2 & 1/2 & 0 \\ 0 & 1/2 & -1 & 3/2 \\ 0 & 0 & 1/2 & -3/2 \\ 0 & 0 & 0 & 1/2 \end{bmatrix} \]
so $L^{-1}(1) = \frac{1}{2}$, $L^{-1}(x) = \frac{1}{2}(x - 1)$, $L^{-1}(x^2) = \frac{1}{2}(x^2 - 2x + 1)$ and $L^{-1}(x^3) = \frac{1}{2}(x^3 - 3x^2 + 3x)$.

5. The eigenvalues of the matrix $A = \begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix}$ are $\lambda = 7$ and $\lambda = 3$ (trust me!).

(a) Find the general solution of the linear system of differential equations:
\[ \frac{dy_1}{dt} = 7y_1 \]
\[ \frac{dy_2}{dt} = 3y_2 \]
This should be easy because the system is “uncoupled”.

(b) Find a matrix \( C \) that diagonalizes \( A \), so that \( C^{-1}AC = \begin{bmatrix} 7 & 0 \\ 0 & 3 \end{bmatrix} \).

(c) Now find the general solution of the “coupled” linear system of differential equations:

\[
\begin{align*}
\frac{dx_1}{dt} &= 5x_1 + 2x_2 \\
\frac{dx_2}{dt} &= 2x_1 + 5x_2
\end{align*}
\]

(a) The solution is \( y_1(t) = c_1e^{7t}, \ y_2(t) = c_2e^{3t} \).

(b) An eigenvector for \( \lambda = 7 \) is \( \begin{bmatrix} 1 \\ 1 \end{bmatrix} \). Likewise an eigenvector for \( \lambda = 3 \) is \( \begin{bmatrix} -1 \\ 1 \end{bmatrix} \). So the matrix \( C \) is

\[
C = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.
\]

(c) Let \( \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = C \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} y_1 - y_2 \\ y_1 + y_2 \end{bmatrix} \), so that \( \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = C^{-1} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(x_1 + x_2) \\ \frac{1}{2}(-x_1 + x_2) \end{bmatrix} \).

Then we’ll have

\[
y_1' = \frac{1}{2}(x_1' + x_2') = \frac{1}{2}(5x_1 + 2x_2 + 2x_1 + 5x_2) = \frac{7}{2}(x_1 + x_2) = 7y_1
\]

and

\[
y_2' = \frac{1}{2}(-x_1' + x_2') = \frac{1}{2}(-5x_1 - 2x_2 + 2x_1 + 5x_2) = \frac{3}{2}(-x_1 + x_2) = 3y_2.
\]

Therefore, \( y_1 = c_1e^{7t} \) and \( y_2 = c_2e^{3t} \) from part (a). But then

\[
x_1 = \frac{1}{2}(y_1 - y_2) = \frac{1}{2}(c_1e^{7t} - c_2e^{3t})
\]

and

\[
x_2 = \frac{1}{2}(y_1 + y_2) = \frac{1}{2}(c_1e^{7t} + c_2e^{3t}),
\]

or (creating new constants \( k_1 = c_1/2 \) and \( k_2 = c_2/2 \)),

\[
\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = k_1 e^{7t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + k_2 e^{3t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}.
\]

6. Let \( V \) be the vector space of all real sequences \((a_1, a_2, a_3, \ldots)\).

(a) Let \( S : V \to V \) be the map that sends \((a_1, a_2, a_3, \ldots)\) to \((a_2, a_3, a_4, \ldots)\). Show that \( S \) is a linear map.

(b) What is the kernel of \( S \)?

(c) What is the image of \( S \)?
7. Let \( A \) be an \( n \times n \) real matrix such that \( A^2 = I \).

(a) Prove that if \( \lambda \) is an eigenvalue of \( A \), then \( \lambda = \pm 1 \).

(b) Let \( \mathbf{v} \) be any non-zero vector in \( \mathbb{R}^n \). Show that \( \mathbf{v} \) can be written as \( \mathbf{v} = \mathbf{w} + \mathbf{x} \) where \( \mathbf{w} \) satisfies \( A\mathbf{w} = \mathbf{w} \) and \( \mathbf{x} \) satisfies \( A\mathbf{x} = -\mathbf{x} \) (it is possible that one of \( \mathbf{w} \) or \( \mathbf{x} \) can be zero). (Hint: consider the two vectors \( \mathbf{v} \) and \( A\mathbf{v} \) and what \( A \) does to the two-dimensional subspace they span if \( A\mathbf{v} \neq \pm \mathbf{v} \).)

(c) (Extra credit) Show that \( A \) is diagonalizable.

(a) Suppose \( A\mathbf{v} = \lambda \mathbf{v} \). Then \( \mathbf{v} = A^2\mathbf{v} = \lambda^2\mathbf{v} \), and so \( \lambda^2 = 1 \) or \( \lambda = \pm 1 \).

(b) Let \( \mathbf{v} \) be any non-zero vector, and let \( \mathbf{w} = \frac{1}{2}(\mathbf{v} + A\mathbf{v}) \) and \( \mathbf{x} = \frac{1}{2}(\mathbf{v} - A\mathbf{v}) \). Then \( \mathbf{w} + \mathbf{x} = \mathbf{v} \) and

\[
A\mathbf{w} = \frac{1}{2}A(\mathbf{v} + A\mathbf{v}) = \frac{1}{2}A\mathbf{v} + \frac{1}{2}A^2\mathbf{v} = \frac{1}{2}(A\mathbf{v} + \mathbf{v}) = \mathbf{w}
\]

and

\[
A\mathbf{x} = \frac{1}{2}A(\mathbf{v} - A\mathbf{v}) = \frac{1}{2}A\mathbf{v} - \frac{1}{2}A^2\mathbf{v} = \frac{1}{2}(A\mathbf{v} - \mathbf{v}) = -\mathbf{x}.
\]

(c) Let \( E(1) \) be the subspace of \( \mathbb{R}^n \) consisting of vectors \( \mathbf{v} \) for which \( A\mathbf{v} = \mathbf{v} \) and let \( E(-1) \) be the subspace of \( \mathbb{R}^n \) consisting of vectors \( \mathbf{w} \) for which \( A\mathbf{w} = -\mathbf{w} \). Let \( \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\} \) be a basis for
Let \( E(1) \) and \( \{w_1, w_2, \ldots, w_\ell\} \) be a basis for \( E(-1) \). In part (b), we showed that every vector in \( \mathbb{R}^n \) can be expressed as the sum of a vector from \( E(1) \) and a vector from \( E(-1) \), therefore the union of the two bases \( \{v_1, v_2, \ldots, v_k, w_1, w_2, \ldots, w_\ell\} \) spans \( \mathbb{R}^n \). Therefore \( k + \ell = n \) and this is a basis of \( \mathbb{R}^n \) consisting of eigenvectors of \( A \). Therefore \( A \) is diagonalizable (and is in fact diagonal with respect to this basis).