Determinants

Determinants are important both from a theoretical and a practical point of view. They provide a way of summarizing and simplifying the information contained in a square matrix.

You have already encountered determinants of 2-by-2 and 3-by-3 matrices:

\[
\begin{vmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{vmatrix} = a_{11}a_{22} - a_{12}a_{21},
\]

and

\[
\begin{vmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{vmatrix} = a_{11}\begin{vmatrix}a_{22} & a_{23} \\a_{32} & a_{33}\end{vmatrix} - a_{12}\begin{vmatrix}a_{21} & a_{23} \\a_{31} & a_{33}\end{vmatrix} + a_{13}\begin{vmatrix}a_{21} & a_{22} \\a_{31} & a_{32}\end{vmatrix}
\]

\[= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} + a_{12}a_{23}a_{31} - a_{12}a_{21}a_{33} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}\]

1. Explain the geometric meaning of these determinants in terms of areas and volumes. What about the sign of the determinant?

We are going to generalize the definition of the determinant to cover all square matrices, no matter how big. We’ll begin by identifying the algebraic properties which characterize the determinant. For this purpose, we use the following notation: If \(a_1, a_2, \ldots, a_n\) are vectors in \(\mathbb{R}^n\), then we will write

\[
[a_1, a_2, \ldots, a_n]
\]

for the matrix whose rows are \(a_1, a_2, \text{ etc.}..\)

2. Show that 2-by-2 determinants and 3-by-3 determinants satisfy the following properties:

(a) Multilinearity: For (appropriately-sized) vectors \(a_1, a_2, a_3, b_1, b_2\) and scalars \(c_1, c_2\) we have

\[
\det[c_1b_1 + c_2b_2, a_2] = c_1 \det[b_1, a_2] + c_2 \det[b_2, a_2],
\]

and

\[
\det[a_1, c_1b_1 + c_2b_2] = c_1 \det[a_1, b_1] + c_2 \det[a_1, b_2]
\]

for 2-by-2 determinants and

\[
\det[c_1b_1 + c_2b_2, a_2, a_3] = c_1 \det[b_1, a_2, a_3] + c_2 \det[b_2, a_2, a_3]
\]

\[
\det[a_1, c_1b_1 + c_2b_2, a_3] = c_1 \det[a_1, b_1, a_3] + c_2 \det[a_1, b_2, a_3]
\]

\[
\det[a_1, a_2, c_1b_1 + c_2b_2] = c_1 \det[a_1, a_2, b_1, a_3] + c_2 \det[a_1, a_2, b_2]
\]

for 3-by-3 determinants.

(b) Alternating: If two rows in a 2-by-2 or 3-by-3 matrix are identical, then the determinant is zero.

(c) Normalization: The determinants of the 2-by-2 and 3-by-3 identity matrices are equal to 1.

We are going to use these three properties to generalize the determinant to the case of \(n\)-by-\(n\) matrices.
**Definition:** A determinant function on $n$-by-$n$ matrices is a function that maps each $n$-by-$n$ matrix $A$ to a scalar called $\det A$ or $|A|$, and which satisfies the following properties:

(a) **Multilinearity:** The determinant function is linear in each row of the matrix, in other words for vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n, \mathbf{b}_1, \mathbf{b}_2$ and scalars $c_1, c_2$, we have

$$\det[c_1 \mathbf{a}_1, \ldots, \mathbf{a}_{i-1}, c_1 \mathbf{b}_1 + c_2 \mathbf{b}_2, \mathbf{a}_{i+1}, \ldots, \mathbf{a}_n] = c_1 \det[c_1 \mathbf{a}_1, \ldots, \mathbf{a}_{i-1}, \mathbf{b}_1, \mathbf{a}_{i+1}, \ldots, \mathbf{a}_n] + c_2 \det[c_1 \mathbf{a}_1, \ldots, \mathbf{a}_{i-1}, \mathbf{b}_2, \mathbf{a}_{i+1}, \ldots, \mathbf{a}_n]$$

for $1 \leq i \leq n$.

(b) **Alternating:** If two rows of the matrix $A$ are identical, then $\det A = 0$.

(c) **Normalization:** If $I_n$ is the $n$-by-$n$ identity matrix, then $\det I_n = 1$.

We will show that determinant functions exist for each $n$, and that there is only one such function for each $n$.

3. A determinant function also satisfies the following properties:

(a) If a row of $A$ consists of the zero vector, then $\det A = 0$.

(b) The determinant changes sign if two rows are interchanged.

(c) The determinant is unchanged if a multiple of one row is added to another row (row operation 3).

(d) The determinant vanishes if the rows are linearly dependent.

4. Using only the definition above and the results of problem 3, prove that the determinants of 2-by-2 matrices and 3-by-3 matrices are given by the formulas at the top of page 1.

5. Prove that the determinant of a diagonal matrix (i.e., a matrix for which $a_{ij} = 0$ if $i \neq j$) is equal to the product of its diagonal elements (i.e., $\det A = a_{11}a_{22} \cdots a_{nn}$).

6. Prove that the determinant of an upper triangular matrix $U$ (i.e., a matrix for which $u_{ij} = 0$ if $i > j$) is equal to the product of its diagonal elements. Do not use the normalization axiom until the last possible moment, so that the next-to-last step in your proof shows that

$$\det U = u_{11}u_{22} \cdots u_{nn} \det(I_n).$$

7. Explain how to calculate the determinant of any square matrix by reducing it to row-echelon form (which is necessarily upper-triangular) and keeping track of how many times you use row operations 1 and 2 during the process.

8. Show that problems 6 and 7 imply that the determinant function is unique, since together they imply that any alternating multilinear scalar-valued function $f$ can be computed as $f(A) = (a$ number that can be computed from $A$) times $f(I)$, so the difference between the determinant and any normalized alternating multilinear function must be zero. Therefore, every alternating multilinear scalar-valued function $f$ has the form $f(A) = \det(A) f(I)$.
9. Prove that \( \det(AB) = \det(A) \det(B) \). To do this, show that, if \( B \) is fixed, then \( f(A) = \det(AB) \) is an alternating multilinear scalar-valued function, and that \( f(I) = \det(B) \).

10. Show that, if \( A \) is an \( n \times n \) invertible matrix, then

\[
\det(A^{-1}) = \frac{1}{\det(A)}.
\]

11. Prove that a set of \( n \) vectors \( \{a_1, a_2, \ldots, a_n\} \) in \( \mathbb{R}^n \) (or \( \mathbb{C}^n \)) is linearly independent if and only if the determinant of the matrix \( [a_1, a_2, \ldots, a_n] \) is not zero.

Next, we turn to a way of calculating determinants called expansion by minors (or cofactor expansion). We begin with a few odd observations:

12. (a) Let \( a_1, a_2, \ldots, a_k \) be vectors in \( \mathbb{R}^k \), and for \( n > k \), let \( T: \mathbb{R}^k \to \mathbb{R}^n \) be the map that sends the vector \([x_1, x_2, \ldots, x_k]\) to the vector \([x_1, x_2, \ldots, x_k, 0, \ldots, 0]\) so \( T \) appends \( n - k \) zeros onto the end of the vector (in linear transformation language, if \( \{e_1, e_2, \ldots, e_k\} \) is a basis for \( \mathbb{R}^k \) and \( \{f_1, f_2, \ldots, f_n\} \) is a basis for \( \mathbb{R}^n \) then \( T(e_i) = f_i \) for \( i = 1, \ldots, k \) determines the map). Prove that

\[
\det[a_1, a_2, \ldots, a_k] = \det[a_1, a_2, \ldots, a_k, f_{k+1}, \ldots, f_n]
\]

in other words:

\[
\det \begin{bmatrix} A & 0_k \times (n-k) \\ 0_{(n-k) \times k} & I_{(n-k) \times (n-k)} \end{bmatrix} = \det(A).
\]

Hint: Show that the expression on the left side of the equals sign satisfies all the axioms of the determinant function.

(b) Likewise, if \( b_1, b_2, \ldots, b_\ell \) are vectors in \( \mathbb{R}^\ell \) and for \( n > \ell \) the map \( T: \mathbb{R}^\ell \to \mathbb{R}^n \) sends \([y_1, \ldots, y_\ell]\) to \([0, \ldots, 0, y_1, \ldots, y_\ell]\) so \( T \) appends \( n - \ell \) zeros to the beginning of the vector (in linear transformation language, \( T(e_i) = f_{(n-\ell)+i} \), prove that

\[
\det[b_1, b_2, \ldots, b_\ell] = \det[f_1, \ldots, f_{n-\ell}, T(e_1), \ldots, T(e_\ell)]
\]

in other words:

\[
\det \begin{bmatrix} I_{(n-\ell) \times (n-\ell)} & 0_{(n-\ell) \times \ell} \\ 0_{(n-\ell) \times \ell} & B \end{bmatrix} = \det(B).
\]

(c) Using (a) and (b), prove the block determinant formula:

\[
\det \begin{bmatrix} A & 0_{k \times \ell} \\ 0_{\ell \times k} & B \end{bmatrix} = \det(A) \det(B)
\]

where \( A \) is a \( k \times k \) matrix and \( B \) is an \( \ell \times \ell \) matrix.

13. Show that if the determinant of a matrix changes sign if two columns are interchanged. (Hint: Let \( A' \) be the result of switching two columns of the matrix \( A \). Show that the function \( \det A' \) is multilinear and alternating in the rows of \( A \) [not \( A' \)] and so by problem 8 we have \( \det A' = (\det P')(\det A) \). Then show that \( \det P' = -1 \).)
If $A$ is an $n$-by-$n$ matrix, then the $ij$th minor of $A$ is the $(n-1)$-by-$(n-1)$ matrix obtained by deleting the $i$th row and the $j$th column of $A$. Let $A'_{ij}$ denote the $ij$th minor of $A$. The determinant of $A'_{ij}$ is called the $ij$th cofactor of $A$.

14. Let $A$ be an $n$-by-$n$ matrix, and let $\widetilde{A}_{ij}$ be the matrix that has a 1 in the $ij$th place, and zeros across the rest of row $i$ and zeros up and down the rest of column $j$, but the rest of $A$ is unchanged. So we have

$$(\widetilde{A}_{ij})_{k\ell} = \begin{cases} 1 & \text{if } k = i \text{ and } \ell = j \\ 0 & \text{if } k = i \text{ and } \ell \neq j \\ 0 & \text{if } k \neq i \text{ and } \ell = j \\ a_{k\ell} & \text{if } k \neq i \text{ and } \ell \neq j \end{cases}$$

(a) Show that $\det \widetilde{A}_{ij} = (-1)^{i+j} \det A'_{ij}$.

(b) Use the linearity of the determinant in row $i$ to show that

$$\det A = \sum_{j=1}^{n} a_{ij} \det \widetilde{A}_{ij}.$$  

(c) Show that

$$\det A = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det A'_{ij}$$

for any $i = 1, \ldots, n$. This is called the cofactor expansion of the determinant of $A$ (along the $i$th row).

(d) Show that the determinant is multilinear in the columns of $A$. Use this, together with problem 13, to show that you can expand the determinant of $A$ by cofactors of its $j$th column, in other words

$$\det A = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det A'_{ij}$$

for any $j = 1, \ldots, n$.

15. The transpose of a matrix $A$ is the matrix $A^t$ whose rows are the same as the columns of $A$. Show that for any $n$-by-$n$ matrix $A$, we have $\det A^t = \det A$.

16. Let $\text{cof} A$ be the matrix of cofactors of $A$ — so $(\text{cof} A)_{ij} = \det A'_{ij}$.

(a) Show that $A(\text{cof} A)^t = (\det A)I$.

(b) Show that if $\det A \neq 0$, then $A^{-1} = \frac{1}{\det A} (\text{cof} A)^t$.

(c) Show that a square matrix $A$ is invertible (non-singular) if and only if $\det A \neq 0$.

17. Let $A$ be an $n$-by-$n$ matrix, Prove that the following are equivalent:

(a) $A$ is invertible.
(b) The only solution of \( Ax = 0 \) is \( x = 0 \).

(c) For any vector \( b \), the equation \( Ax = b \) has at most one solution.

(d) For any vector \( b \), the equation \( Ax = b \) has at least one solution.

(e) The columns of \( A \) are linearly independent.

(f) The columns of \( A \) span \( \mathbb{R}^n \).

(g) The rows of \( A \) are linearly independent.

(h) The rows of \( A \) span \( \mathbb{R}^n \).

(e) \( \det A \neq 0 \).

The result of problem 17 is central in linear algebra.

18. The 3-by-3 Vandermonde determinant is

\[
V_3 = \begin{vmatrix}
1 & 1 & 1 \\
a & b & c \\
a^2 & b^2 & c^2 \\
\end{vmatrix}
\]

(a) Show that \( V_3 = (a - b)(b - c)(c - a) \).

(b) Generalize this to the \( n \)-by-\( n \) Vandermonde (be clever — the determinant is a polynomial; what is its degree? what are its zeros?).

19. (a) Show that the equation of the line through the two points \( (a, b) \) and \( (c, d) \) is

\[
\det \begin{bmatrix}
1 & x & y \\
1 & a & b \\
1 & c & d \\
\end{bmatrix} = 0.
\]

(b) Show that the equation of the parabola through the three points \( (a, b) \), \( (c, d) \) and \( (e, f) \) (with \( a, c \) and \( e \) all distinct) is

\[
\det \begin{bmatrix}
1 & x & x^2 & y \\
1 & a & a^2 & b \\
1 & c & c^2 & d \\
1 & e & e^2 & f \\
\end{bmatrix} = 0.
\]