Eigenvalues and eigenvectors

In this part of the course, we’re going to be considering linear maps from a vector space to itself, i.e., \( T : V \rightarrow V \). Therefore, when we consider matrix representations of our linear maps they will be square.

We know from problem 25 of the January 10 notes that when \( T : V \rightarrow W \) and \( V \) and \( W \) are different spaces, we can always choose bases for \( V \) and \( W \) so that the matrix of \( T \) is especially simple (it has 1’s and 0’s on the main diagonal and zeros everywhere else).

When we consider mappings from a space to itself, we only get to choose the basis once, so such a simple matrix representation might not be possible. However, as we shall see, for many (even most) mappings \( T \) it will be possible to choose a basis of \( V \) with respect to which the matrix of \( T \) is diagonal. This can provide a huge simplification in many theoretical and applied problems.

Along the way, we will see how to recognize and calculate intrinsic invariants of the linear transformation \( T \). These are quantities that do not depend on the choice of basis in which the matrix for \( T \) is expressed. The determinant is one such invariant, but as we shall see, there are several others. The idea is to gain insight into what the matrix does by looking at a few numbers (i.e., the determinant tells by what factor \( T \) distorts volumes, and whether \( T \) preserves or reverses orientation).

The key concepts for accomplishing all this are the notions of eigenvectors and eigenvalues of a linear transformation (or of a matrix). So we start there:

**Definition.** A nonzero vector \( v \in V \) is called an eigenvector of the linear transformation \( T : V \rightarrow V \) if there is a (possibly zero) scalar \( \lambda \), called the eigenvalue of \( T \) corresponding to \( v \), such that

\[
Tv = \lambda v.
\]

1. Show that for each eigenvector \( v \), there is only one eigenvalue \( \lambda \).

2. What are the eigenvalues and eigenvalues of the identity operator? . . . of the zero operator? . . . of the operator that multiplies every vector \( v \in V \) by a fixed scalar \( c \)?

3. Show that the eigenvalues of a diagonal matrix are the diagonal entries. What are the eigenvectors?

4. Conversely to problem 3, suppose that \( V \) is finite-dimensional and there is a basis \( \{v_1, v_2, \ldots, v_n\} \) of \( V \) of eigenvectors of \( T \) whose respective eigenvalues are \( \lambda_1, \lambda_2, \ldots, \lambda_n \) (the \( \lambda_i \)'s need not all be different). What is the matrix representation of \( T \) with respect to this basis. Explain why we say that this basis diagonalizes \( T \).

5. We say that \( \lambda \) is an eigenvalue of \( T \) if there is at least one eigenvector \( v \) having eigenvalue \( \lambda \). Suppose \( \lambda \) is an eigenvalue of \( T \) and let \( E(\lambda) \) be the set of all vectors \( v \) (including \( v = 0 \)) such that \( Tv = \lambda v \) for this value of \( \lambda \). Show that \( E(\lambda) \) is a subspace of \( V \).
6. If zero is an eigenvalue of $T$, what is $E(0)$? What can you say about $T$ in this case?

7. What are the eigenvalues and eigenvectors of the map $T: \mathbb{R}^2 \to \mathbb{R}^2$ that reflects every vector through the $x$-axis (so $T(x, y) = (x, -y)$)?

8. Show that if you consider the plane as $\mathbb{R}^2$, then the map that rotates vectors an angle of $\pi/4$ has no eigenvectors. What if you consider the plane as $\mathbb{C}^1$?

9. What are the eigenvectors (eigenfunctions?) of the derivative operator $D : C^\infty \to C^\infty$? How about of the integration operator defined by $T(f) = g$, where

$$g(x) = \int_0^x f(t) \, dt$$

10. Prove that if $v_1$ is an eigenvector of $T$ corresponding to the eigenvalue $\lambda_1$ and $v_2$ is an eigenvector corresponding to $\lambda_2$, and if $\lambda_1 \neq \lambda_2$ then $v_1$ and $v_2$ are linearly independent. Extend this to the case of several eigenvectors whose eigenvalues are all distinct.

11. Show that if $\dim V = n$, then a linear transformation $T: V \to V$ can have at most $n$ distinct eigenvalues. Also, if $T$ does have $n$ distinct eigenvalues, there is a basis with respect to which the matrix of $T$ is diagonal.

Calculating eigenvalues and eigenvectors

If $\dim V$ is finite, then we can use determinants and row reduction to calculate the eigenvalues and eigenvectors of $T: V \to V$.

12. Prove that if $v$ is an eigenvector of $T$ with eigenvalue $\lambda$, then the kernel of the map $T - \lambda I$ (where $I$ is the identity map) has dimension at least 1.

13. If $A$ is the matrix of the linear transformation $T$ with respect to some basis, and if $\lambda$ is an eigenvalue of $T$, then $\det(A - \lambda I) = 0$.

The equation $\det(A - \lambda I) = 0$ is a polynomial equation of degree $n$ in $\lambda$. The polynomial $\det(A - \lambda I)$ is called the characteristic polynomial of $A$ and the equation $\det(A - \lambda I) = 0$ is called the characteristic equation of $A$.

14. Find the eigenvalues of the matrix

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 2 & 1 & 1 \\ -1 & -1 & -2 \end{bmatrix}$$

and their corresponding eigenvectors.

15. Find the eigenvalues of the matrix

$$A = \begin{bmatrix} 2 & 0 & 2 \\ -1 & 3 & 1 \\ 1 & -1 & 3 \end{bmatrix}$$
and their corresponding spaces of eigenvectors.

For each eigenvalue \( \lambda \), the order of \( \lambda \) as a root of the characteristic polynomial (i.e., the number of times \( x - \lambda \) occurs as a factor of the characteristic polynomial) is called the **algebraic multiplicity** of \( \lambda \). The dimension of \( E(\lambda) \) is called the **geometric multiplicity** of \( \lambda \). As you can see from the preceding problem, sometimes these are different.

16. Find the eigenvalues of the matrix

\[
A = \begin{bmatrix}
3 & 3 & 4 \\
2 & 3 & 2 \\
2 & 1 & 1
\end{bmatrix}
\]

and bases for their corresponding spaces of eigenvectors.

17. Show that the product of the eigenvalues of \( A \) (counted with their algebraic multiplicities) is equal to the determinant of \( A \).

18. Show that the sum of the eigenvalues of \( A \) (counted with their algebraic multiplicities) is equal to the trace of \( A \) (which is the sum \( \sum a_{ii} \) of the diagonal elements of \( A \)).

It is an important fact that if \( T: V \to V \) is a linear transformation and \( V \) is finite-dimensional, then the matrix representations of \( T \) with respect to any two bases have the same characteristic polynomial.

Two matrices that represent the same linear transformation with respect to two different bases are called **similar**. The next few problems explore this concept.

19. Let \( \{e_1, e_2, \ldots, e_n\} \) and \( \{f_1, f_2, \ldots, f_n\} \) be two bases of the vector space \( V \), and let \( A \) be the matrix of the linear transformation \( T: V \to V \) with respect to the \( e \) basis, and let \( B \) be the matrix of the \( T \) with respect to the second basis.

(a) Show that each vector \( f_i \) can be expressed as a linear combination of the \( e_j \)'s:

\[
f_i = s_{1i}e_1 + s_{2i}e_2 + \cdots + s_{ni}e_n.
\]

(Note carefully the order of the subscripts!)

(b) If \( v \in V \), and \( v = c_1f_1 + \cdots + c_nf_n \), show that we also have \( v = d_1e_1 + \cdots + d_ne_n \), where

\[
\begin{bmatrix}
d_1 \\
d_2 \\
\vdots \\
d_n
\end{bmatrix} =
\begin{bmatrix}
s_{11} & s_{12} & \cdots & s_{1n} \\
s_{21} & s_{22} & \cdots & s_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
s_{n1} & s_{n2} & \cdots & s_{nn}
\end{bmatrix}
\begin{bmatrix}
c_1 \\
c_2 \\
\vdots \\
c_n
\end{bmatrix}.
\]

(Thus, the \( i \)th column of the matrix \( S \) contains the coefficients of the vector \( f_i \) expressed in terms of the \( e \) basis.)

(c) Show that \( S^{-1} \) gives the way to translate the coefficients of a vector given in terms of the \( e \) basis to the coefficients of the vector with respect to the \( f \) basis.
(d) The matrix $A$ takes a vector $v$ expressed in terms of the $e$ basis and produces the coefficients of the vector $T(v)$ expressed in terms of the $e$ basis. Likewise the matrix $B$ takes the vector $v$ expressed in terms of the $f$ basis and produces the coefficients of $T(v)$ expressed in terms of the $f$ basis. Show that $B = S^{-1}AS$.

If $B = SAS^{-1}$ for some invertible matrix $S$, then $B$ and $A$ are called similar matrices. As illustrated in problem 19, similar matrices represent the same linear transformation with respect to different bases.

20. (a) Explain, from the point of view of linear transformations, why similar matrices should have the same eigenvalues.

(b) Show that if $A$ and $B$ are similar matrices, then $A$ and $B$ have the same determinant, the same trace (to do this, it might help to show first that $\text{tr}(MN) = \text{tr}(NM)$ for any $n$-by-$n$ matrices $M$ and $N$), and the same characteristic polynomial.