To discuss in recitation:

1. Prove Lemma 4, using the proof of Lemma 2 as a guide.

2. Prove that the first few terms of a sequence do not affect convergence (or the limit). More formally, show that if there exists an \( N \) such that \( a_n = b_n \) for all \( n \geq N \), then \( a_n \to L \) as \( n \to \infty \) implies \( b_n \to L \) as \( n \to \infty \).

3. Show by means of an example that if \( a_n \to L \) and \( a_n > M \) for all \( n \), it does not necessarily follow that \( L \geq M \) (i.e., taking the limit can destroy strict inequality).

   Does it follow that \( L \geq M \)? Proof or counterexample.

4. (a) Show from the definition that if \( a_n \to L \), then \( ca_n \to cL \) (as \( n \to \infty \)).

   (b) Show from the definition (don’t use Lemma 2) that if \( a_n \to L \) as \( n \to \infty \), then \( a_n^2 \to L^2 \).

5. Use Theorem 8 to prove: If \( f: [a, b] \to \mathbb{R} \) is continuous and \( f(a) \geq C \geq f(b) \), then there exists \( c \in [a, b] \) such that \( f(c) = C \).

To be handed in on September 6:

1. (a) Suppose \( a_n \geq c_n \geq b_n \) for all \( n \). Show that, if \( a_n \to L \) and \( b_n \to L \) as \( n \to \infty \) then also \( c_n \to L \) as \( n \to \infty \).

   (b) Now suppose \( |a_n| \geq |c_n| \geq |b_n| \) for all \( n \) and that \( a_n \to L \) and \( b_n \to L \) as \( n \to \infty \). Does it follow that \( c_n \to L \) as \( n \to \infty \)? Proof or counterexample.

   (a) Given \( \varepsilon > 0 \), we know that there is an \( N_a \) so that \( |a_n - L| < \varepsilon \) provided \( n > N_a \), and there is an \( N_b \) so that \( |b_n - L| < \varepsilon \) for all \( n > N_b \). We can rewrite \( |a_n - L| < \varepsilon \) as
   
   \[-\varepsilon < a_n - L < \varepsilon \]
   
   or
   
   \[L - \varepsilon < a_n < L + \varepsilon \text{ for } n > N_a \]
   
   and similarly
   
   \[L - \varepsilon < b_n - L < \varepsilon \text{ for } n > N_b.\]
   
   Let \( N_c \) be the larger of \( N_a \) and \( N_b \). Then for \( n > N_c \) we have
   
   \[L - \varepsilon < b_n \leq c_n \leq a_n < L + \varepsilon, \]
so

\[-\varepsilon < c - L < \varepsilon\]

which shows that \(c_n \to L\) as \(n \to \infty\).

(b) Let \(a_n = \frac{n + 1}{n}\), let \(b_n = \frac{n}{n + 1}\) and let \(c_n = (-1)^n\). Then \(|a_n| \geq |c_n| \geq |b_n|\) but the sequence \(\{c_n\}\) doesn’t even converge.

2. Show that any real polynomial of odd degree has at least one root. Is the result true for polynomials of even degree? Proof or counterexample.

3. (a) Suppose that \(g: [0, 1] \to [0, 1]\) is a continuous function. Show that there exists a \(c \in [0, 1]\) with \(g(c) = c\) (i.e., every continuous map of \([0, 1]\) to itself has a “fixed point”). (Hint: consider \(f(x) = g(x) - x\))

(b) Give an example of a bijective (one-to-one and onto) function \(h: (0, 1) \to (0, 1)\) such that \(h(x) \neq x\) for all \(x \in (0, 1)\).

This exercise shows that there is an essential difference between open and closed intervals.

4. (a) Suppose \(g: (A, B) \to \mathbb{R}\) is a differentiable function such that \(g'(x) \geq 0\) for all \(x \in (A, B)\). For \(a, b \in (A, B)\) with \(b > a\), show that \(g(b) - g(a) \geq 0\).

(b) Prove the converse of the result in part (a) (you might want to prove that \(g'(x) \geq -\varepsilon\) for all \(\varepsilon > 0\)).

5. Assume that \(a_n \geq 1\) for all \(n\), and that \(a_n + a_n^{-1}\) tends to a limit as \(n \to \infty\). Show that \(a_n\) also tends to a limit.

By giving an example, show that the result is false if we replace the condition \(a_n \geq 1\) with \(a_n \geq k\) where \(0 < k < 1\).

Let \(b_n = a_n + a_n^{-1}\). Then \(a_n b_n = a_n^2 + 1\), so

\[a_n = \frac{b \pm \sqrt{b^2 - 4}}{2}\]

If we assume that \(a_n \geq 1\), then we must choose the plus sign so

\[a_n = \frac{b + \sqrt{b^2 - 4}}{2}\]

The rules for limits in Lemma 2 in the notes almost show that if \(b_n \to L\) as \(n \to \infty\), then \(a_n\) converges to \(\frac{1}{2}(L + \sqrt{L^2 - 4})\), except we don’t know that if \(x_n \geq 0\) and \(x_n \to Q\) as \(n \to \infty\) then \(\sqrt{x_n} \to \sqrt{Q}\). We have to prove that.

There are two cases: either \(Q = 0\) or \(Q > 0\). For the first case, given \(\varepsilon > 0\) we choose \(N\) so that \(x_n ^2 < \varepsilon^2\) provided \(n > N\). Then we’ll have \(\sqrt{x_n^2} < \varepsilon\) for \(n > N\) and this shows that \(\sqrt{x_n^2} \to 0\) as \(n \to \infty\).
In the second case, given \( \varepsilon > 0 \), choose \( N \) so that \( |x_n - Q| < \frac{3}{2}\sqrt{Q} \varepsilon \) and \( |x_n - Q| < \frac{3}{4}Q \). The latter condition guarantees that \( x_n > \frac{1}{4}Q \) for \( n > N \), and so \( \sqrt{x_n} > \frac{1}{2}\sqrt{Q} \) for \( n > N \). Then, because 
\[
x_n - Q = (\sqrt{x_n} - \sqrt{Q})(\sqrt{x_n} + \sqrt{Q})
\] we have (for \( n > N \))
\[
|\sqrt{x_n} - \sqrt{Q}| = \frac{|x_n - Q|}{\sqrt{x_n} + \sqrt{Q}} < \frac{|x_n - Q|}{\frac{1}{2}\sqrt{Q} + \sqrt{Q}} = \frac{2|x_n - Q|}{3\sqrt{Q}} < \varepsilon
\]
which shows that \( \sqrt{x_n} \to \sqrt{Q} \) as \( n \to \infty \).