MATH 360 –Homework 1

Due Wednesday, September 6, 2017

To discuss in recitation:

1. Prove Lemma 4, using the proof of Lemma 2 as a guide.

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2. Prove that the first few terms of a sequence do not affect convergence (or the limit). More formally, show that if there exists an N such that $a_n = b_n$ for all $n \geqslant N$, then $a_n \to L$ as $n \to \infty$ implies $b_n \to L$ as $n \to \infty$.

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3. Show by means of an example that if $a_n \to L$ and $a_n > M$ for all n, it does not necessarily follow that $L > M$ (i.e., taking the limit can destroy strict inequality).

Does it follow that $L \geqslant M$? Proof or counterexample.

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4. (a) Show from the definition that if $a_n \to L$, then $ca_n \to cL$ (as $n \to \infty$).

(b) Show from the definition (don't use Lemma 2) that if $a_n \to L$ as $n \to \infty$, then $a_n^2 \to L^2$.

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5. Use Theorem 8 to prove: If $f\colon [a,b] \to \mathbb{R}$ is continuous and $f(a) \geqslant C \geqslant f(b)$, then there exists $c \in [a,b]$ such that $f(c) = C$.

To be handed in on September 6:

1. (a) Suppose $a_n \geqslant c_n \geqslant b_n$ for all n. Show that, if $a_n \to L$ and $b_n \to L$ as $n \to \infty$ then also $c_n \to L$ as $n \to \infty$.

(b) Now suppose $|a_n| \geqslant |c_n| \geqslant |b_n|$ for all n and that $a_n \to L$ and $b_n \to L$ as $n \to \infty$. Does it follow that $c_n \to L$ as $n \to \infty$? Proof or counterexample.

(a) Given $\varepsilon > 0$, we know that there is an N_a so that $|a_n - L| < \varepsilon$ provided $n > N_a$, and there is an N_b so that $|b_n - L| < \varepsilon$ for all $n > N_b$. We can rewrite $|a_n - L| < \varepsilon$ as

$$-\varepsilon < a_n - L < \varepsilon$$

or

$$L - \varepsilon < a_n < L + \varepsilon \quad \text{for } n > N_a$$

and similarly

$$L - \varepsilon < b_< L + \varepsilon \quad \text{for } n > N_b.$$

Let N_c be the larger of N_a and N_b. Then for $n > N_c$ we have

$$L - \varepsilon < b_n \leqslant c_n \leqslant a_n < L + \varepsilon,$$

so

$$-\varepsilon < c - L < \varepsilon$$

which shows that $c_n \to L$ as $n \to \infty$.

(b) Let $a_n = \dfrac{n+1}{n}$, let $b_n = \dfrac{n}{n+1}$ and let $c_n = (-1)^n$. Then $|a_n| \geqslant |c_n| \geqslant |b_n|$ but the sequence $\{c_n\}$ doesn't even converge.

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2. Show that any real polynomial of odd degree has at least one root. Is the result true for polynomials of even degree? Proof or counterexample.

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3. (a) Suppose that $g\colon [0,1] \to [0,1]$ is a continuous function. Show that there exists a $c \in [0,1]$ with $g(c) = c$ (i.e., every continuous map of $[0,1]$ to itself has a "fixed point"). (*Hint*: consider $f(x) = g(x) - x$)

(b) Give an example of a bijective (one-to-one and onto) function $h\colon (0,1) \to (0,1)$ such that $h(x) \neq x$ for all $x \in (0,1)$.

This exercise shows that there is an essential difference between open and closed intervals.

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4. (a) Suppose $g\colon (A,B) \to \mathbb{R}$ is a differentiable function such that $g'(x) \geqslant 0$ for all $x \in (A,B)$. For $a, b \in (A,B)$ with $b > a$, show that $g(b) - g(a) \geqslant 0$.

(b) Prove the converse of the result it part (a) (you might want to prove that $g'(x) \geqslant -\varepsilon$ for all $\varepsilon > 0$).

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5. Assume that $a_n \geqslant 1$ for all n, and that $a_n + a_n^{-1}$ tends to a limit as $n \to \infty$. Show that a_n also tends to a limit.

By giving an example, show that the result is false if we replace the condition $a_n \geqslant 1$ with $a_n \geqslant k$ where $0 < k < 1$.

Let $b_n = a_n + a_n^{-1}$. Then $a_n b_n = a_n^2 + 1$, so

$$a_n = \frac{b \pm \sqrt{b^2 - 4}}{2}.$$

If we assume that $a_n \geqslant 1$, then we must choose the plus sign so

$$a_n = \frac{b + \sqrt{b^2 - 4}}{2}.$$

The rules for limits in Lemma 2 in the notes almost show that if $b_n \to L$ as $n \to \infty$, then a_n converges to $\frac{1}{2}(L + \sqrt{L^2 - 4})$, except we don't know that if $x_n \geqslant 0$ and $x_n \to Q$ as $n \to \infty$ then $\sqrt{x_n} \to \sqrt{Q}$. We have to prove that.

There are two cases: either $Q = 0$ or $Q > 0$. For the first case, given $\varepsilon > 0$ we choose N so that $x_n < \varepsilon^2$ provided $n > N$. Then we'll have $\sqrt{x^n} < \varepsilon$ for $n > N$ and this shows that $\sqrt{x^n} \to 0$ as $n \to \infty$.

In the second case, given $\varepsilon > 0$, choose N so that $|x_n - Q| < \frac{3}{2}\sqrt{Q}\,\varepsilon$ ***and*** $|x_n - Q| < \frac{3}{4}Q$. The latter condition guarantees that $x_n > \frac{1}{4}Q$ for $n > N$, and so $\sqrt{x_n} > \frac{1}{2}\sqrt{Q}$ for $n > N$. Then, because $x_n - Q = (\sqrt{x_n} - \sqrt{Q})(\sqrt{x_n} + \sqrt{Q})$ we have (for $n > N$)

$$|\sqrt{x_n} - \sqrt{Q}| = \frac{|x_n - Q|}{\sqrt{x_n} + \sqrt{Q}} < \frac{|x_n - Q|}{\frac{1}{2}\sqrt{Q} + \sqrt{Q}} = \frac{2|x_n - Q|}{3\sqrt{Q}} < \varepsilon$$

which shows that $\sqrt{x_n} \to \sqrt{Q}$ as $n \to \infty$.
