MATH 360 –Homework 10

Due Wednesday, November 29, 2017

To discuss in recitation:

1. Textbook page 254, problems 1, 3, 4.

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2. Textbook page 262, problems 2, 3, 4.

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3. (a) Express $\int_0^x \arctan t\, dt$ as a power series in x, and discuss the range of validity of this series (endpoints)?

(b) Using (a), show that

$$\frac{\pi}{4} - \ln\sqrt{2} = 1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - - + + \cdots$$

To be handed in on November 29:

1. Textbook page 254, problem 2.

If $f_n(0) = 0$ for all n, so the sequence certainly converges to zero there. For any fixed $x > 0$, we can use L'Hospital's rule to show that (keep in mind the the variable that is changing is n, not x!):

$$\lim_{n\to\infty} \frac{nx}{e^{nx^2}} = \lim_{n\to\infty} \frac{x}{x^2 e^{nx^2}} = \lim_{n\to\infty} \frac{1}{xe^{nx^2}} = 0$$

since $e^{nx^2} \to \infty$ as $n \to \infty$. So the sequence $\{f_n(x)\}$ converges pointwise to zero.
On the other hand (with $u = nx^2$, so $du = 2nx\, dx$):

$$\int_0^1 f_n(x)\, dx = \int_0^1 nxe^{-nx^2}\, dx = \int_0^n \frac{1}{2}e^{-u}\, du = -\frac{1}{2}e^{-u}\Big|_0^n = \frac{1}{2}(1 - e^{-n})$$

which approaches $\frac{1}{2}$ as $n \to \infty$.
This does not contradict Theorem 9.32 because the sequence $\{f_n(x)\}$ does not converge uniformly on $[0, 1]$. To see this, consider

$$f_n'(x) = ne^{-nx^2}(1 - 2nx^2)$$

which is zero when $x = \sqrt{1/(2n)}$, and we compute that the maximum value of $f_n(x)$ on $[0, 1]$ is

$$f\left(\sqrt{\frac{1}{2n}}\right) = \sqrt{\frac{n}{2e}}$$

which goes to ∞ as $n \to \infty$ – so as n gets larger and larger, f_n has a higher and higher peak that is closer and closer to $x = 0$, demonstrating that the convergence cannot be uniform.

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2. Textbook page 262, problems 5, 10.

Problem 5: Since $1 - \frac{1}{x} = \frac{x-1}{x}$, the series is a geometric series with ratio $\frac{x-1}{x}$, which converges if $\left|\frac{x-1}{x}\right| < 1$, which is clearly equivalent to $|x-1| < |x|$, and it converges to

$$\sum_{k=0}^{\infty} \left(\frac{x-1}{x}\right)^k = \frac{1}{1 - \left(1 - \frac{1}{x}\right)} = x.$$

Problem 10: By the ratio test:

$$\lim_{k\to\infty} \left| \frac{x^{2k_2}}{(2k+2)!} \frac{(2k)!}{x^{2k}} \right| = \lim_{k\to\infty} \frac{x^2}{(2k+2)(2k+1)} = 0$$

so the power series $h(x)$ converges (uniformly, as do the series for its derivatives) for all $x \in \mathbb{R}$ and thus may be differentiated term-by-term. The same is true of $g(x)$ as a similar computation shows. We have:

$$h'(x) = \sum_{k=1}^{\infty} \frac{x^{2k-1}}{(2k-1)!} = \sum_{\ell=0}^{\infty} \frac{x^{2\ell+1}}{(2\ell+1)!} = g(x)$$

(the first sum starts at $k = 1$ because the derivative of the $k = 0$ term is zero, and put $k = \ell + 1$ to get the second sum), and

$$g'(x) = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!} = h(x).$$

But then $h''(x) = (g(x))' = h(x)$ and $g''(x) = (h(x))' = g(x)$. Also, notice that $h(0) = 1$ and $g(0) = 0$. Therefore, if $f(x) = \alpha h(x) + \beta g(x)$, we have

$$f''(x) = \alpha h''(x) + \beta g''(x) = \alpha h(x) + \beta g(x) = f(x)$$

so $f'' - f = 0$ and $f(0) = \alpha h(0) + \beta g(0) = \alpha$ and $f'(0) = \alpha h'(0) + \beta g'(0) = \alpha g(0) + \beta h(0) = \beta$.

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3. Textbook page 268, problem 4.

Let $F(x)$ be Weierstrass's continuous but nowhere-differentiable function, and define

$$G(x) = \int_0^x F(t)\,dt.$$

G exists because F, being continuous and bounded, is integrable, and $G'(x) = F(x)$. So G is continuously differentiable. But since F is not differentiable for any x, there is no point where G has a second derivative.

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4. Justify the formula

$$\int_0^1 \frac{t^{p-1}}{1+t^q}\,dt = \frac{1}{p} - \frac{1}{p+q} + \frac{1}{p+2q} - \cdots$$

for positive integers p and q. For $p = 10$ and $q = 40$, use this to estimate the integral to two decimal places.

Using the geometric series formula, we have

$$\frac{1}{1+t^q} = 1 - t^q + t^{2q} + t^{3q} + \cdots ,$$

which converges for $0 \leqslant t < 1$. So, *formally*, we have

$$\begin{aligned}\int_0^1 \frac{t^{p-1}}{1+t^q}\,dt &= \int_0^1 t^{p-1} - t^{p+q-1} + t^{p+2q-1} - t^{p+3q-1} + \cdots\,dt \\ &= \left(\frac{t^p}{p} - \frac{t^{p+q}}{p+q} + \frac{t^{p+2q}}{p+2q} - \frac{t^{p+3q}}{t+3q} + \cdots\right)\Bigg|_0^1 \\ &= \frac{1}{p} - \frac{1}{p+q} + \frac{1}{p+2q} - \frac{1}{p+3q} + \cdots\end{aligned}$$

The last series is convergent by the alternating series test. But the equality between the original integral and the sum of the series has not been justified, because the geometric series does not converge uniformly on $[0,1]$ (in fact, it does not even converge at $t = 1$). So we have to resolve this. Of course, the geometric series (being a power series) does converge on any interval of the form $[0,b]$ as long as $0 < b < 1$. Some notation: Let

$$I(b) = \int_0^b \frac{t^{p-1}}{1+t^q}\,dq \qquad \text{and} \qquad S(b) = \sum_{n=0}^{\infty} \frac{b^{p+nq}}{p+nq}$$

Then by the theorem on integrating uniformly convergent series term-by-term, we are justified in writing $I(b) = S(b)$ provided $0 < b < 1$. Also, because the integrand is a continuous function of t on $[0,1]$, we have that $I(b)$ is a continuous (even differentiable) function of b on the same interval, so the limit $\lim_{b\to 1^-} I(b) = I(1)$ is justified. So all that remains is to show that $\lim_{b\to 1^-} S(b) = S(1)$.

Let

$$P(b,N) = \sum_{n=1}^{N} \frac{b^{p+nq}}{p+nq}$$

be the Nth partial sum of $S(b)$. From the alternating series test, we know that

$$|S(b) - P(b,N)| \leqslant \frac{b^{p+(N+1)q}}{p+(N+1)q} \leqslant \frac{1}{p+(N+1)q}$$

for all $b \leqslant 1$, and this approaches zero as $N \to \infty$. Also, for any fixed N, $P(b,N)$ is a polynomial in b, which is continuous function of b.

Therefore, given $\varepsilon > 0$ we can find an N so that $|S(b) - P(b,N)| < \frac{1}{3}\varepsilon$ and $|S(1) - P(1,N)| < \frac{1}{3}\varepsilon$. And for this N we we can find a $\delta > 0$ such that $|P(b,N) - P(b,1)| < \frac{1}{2}\varepsilon$. Therefore we have

$$|S(b) - S(1)| \leqslant |S(b) - P(b,N)| + |P(b,N) - P(1,N)| + |P(1,N) - S(1)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

and so $\lim_{b\to 1^-} S(b) = S(1)$.

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5. Prove that the series $\displaystyle\sum_{n=1}^{\infty} \frac{2x}{n^2 - x^2}$ is uniformly convergent on any finite closed interval not containing any of the points $\pm 1, \pm 2, \ldots$.

The series clearly does not converge for $x = \pm 1, \pm 2, \ldots$ because the nth term is undefined if $x = \pm n$. So any interval of convergence must avoid these points. On the closed interval $[a, b]$, where $k < a < b < k + 1$ for some $k \in \mathbb{Z}$ (or else $-1 < a < b < 1$ in which case set $k = 1$), then all the terms of the series are defined, and

$$\left| \frac{2x}{n^2 - x^2} \right| < \frac{2b}{n^2 - (|k| + 1)^2}$$

for all $n > |k| + 2$. Since

$$\sum_{n=|k|+3}^{\infty} \frac{2b}{n^2 - (|k| + 1)^2}$$

is a convergent numerical series for fixed b and k, the Weierstrass M-test tells us that the series $\displaystyle\sum_{n=1}^{\infty} \frac{2x}{n^2 - x^2}$ converges uniformly on $[a, b]$.

For the record, it is true that

$$\frac{1}{x} - \pi \cot \pi x = \sum_{n=1}^{\infty} \frac{2x}{n^2 - x^2}.$$