MATH 360 –Homework 2

Due Monday, September 18, 2017

To discuss in recitation (the first two are from Homework 1):

1. Prove Lemma 4, using the proof of Lemma 2 as a guide.

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2. Use Theorem 8 to prove: If $f\colon [a,b] \to \mathbb{R}$ is continuous and $f(a) \geqslant C \geqslant f(b)$, then there exists $c \in [a,b]$ such that $f(c) = C$.

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3. (a) Give an "ε-N" definition of what it means for a sequence ***not*** to converge. Use your definition to show that $x_n = (-1)^n$ is not convergent.

(b) Give an "ε-δ" definition of what it means for a function ***not*** to be continuous at $x = a$. Use your definition to show that

$$f(x) = \begin{cases} \dfrac{x}{|x|} & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad x = 0 \end{cases}$$

is not continuous at $x = 0$.

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4. Suppose the function $f\colon (0,1) \to \mathbb{R}$ is continuous and satisfies $0 < f(x) < x$. Define the sequence of functions $f_n\colon (0,1) \to \mathbb{R}$ by

$$f_1(x) = f(x), \qquad f_n(x) = f(f_{n-1}(x)) \quad \text{for } n \geqslant 2.$$

Prove that $f_n(x) \to 0$ as $n \to \infty$ for all $x \in (0,1)$.

To be handed in on September 18 (the first three are from Homework 1):

1. Show that any real polynomial of odd degree has at least one root. Is the result true for polynomials of even degree? Proof or counterexample.

Write the polynomial as $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, where $a_n \neq 0$ and n is odd. Since $a_n \neq 0$ we can divide the equation $p(x) = 0$ through by a_n and assume that the polynomial is *monic*, i.e., that the leading coefficient is 1. In fact we will assume we have done this, so the polynomial we are working with is

$$m(x) = x^n + \sum_{k=0}^{n-1} b_k x^k.$$

If $|x| > M = n \max\{|b_k| + 1, k = 0 \ldots n-1\}$, then

$$\left| \sum_{k=0}^{n-1} b_k x^k \right| \leqslant \sum_{k=0}^{n-1} |b_k||x|^k < n \max\{|b_k|, k = 0 \ldots n-1\}|x|^{n-1} < |x|^n.$$

Then for positive $x > M$, we have $x^n > 0$ and in fact $m(x) > x^n - x^n = 0$. Likewise for negative $x < -M$ we have $x^n < 0$ (since n is odd) and $m(x) < x^n - x^n = 0$. So we can apply the intermediate value theorem to conclude that there is a c with $-M < c < M$ where $f(c) = 0$.

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2. (a) Suppose that $g\colon [0,1] \to [0,1]$ is a continuous function. Show that there exists a $c \in [0,1]$ with $g(c) = c$ (i.e., every continuous map of $[0,1]$ to itself has a "fixed point"). (*Hint*: consider $f(x) = g(x) - x$)

(b) Give an example of a bijective (one-to-one and onto) function $h\colon (0,1) \to (0,1)$ such that $h(x) \neq x$ for all $x \in (0,1)$.

This exercise shows that there is an essential difference between open and closed intervals.

(a) Following the hint, let $f(x) = g(x) - x$. If $g(0) = 0$ then $c = 0$ is a fixed point. Otherwise, we have $g(0) > 0$, so that $f(0) > 0$. Next, if $g(1) = 1$ then $c = 1$ is a fixed point. Otherwise we have $g(1) < 1$, so that $f(1) < 0$. Therefore (since f is continuous, being the sum of continuous functions), there is a point c in $[0,1]$ where $f(c) = 0$, and so $g(c) = c$.

(b) The function $h(x) = x^2$ is bijective from $(0,1)$ to $(0,1)$ with $h(x) = x^2 \neq x$.

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3. (a) Suppose $g\colon (A,B) \to \mathbb{R}$ is a differentiable function such that $g'(x) \geqslant 0$ for all $x \in (A,B)$. For $a, b \in (A,B)$ with $b > a$, show that $g(b) - g(a) \geqslant 0$.

(b) Prove the converse of the result it part (a) (you might want to prove that $g'(x) \geqslant -\varepsilon$ for all $\varepsilon > 0$).

(a) The mean value inequality says that if $f'(x) \leqslant K$ on $[a,b]$ then $f(b) - f(a) \leqslant K(b-a)$.

Now suppose $f'(x) \geqslant K$ on $[a,b]$. Then the function $h(x) = -f(x)$ satisfies $h'(x) \leqslant -K$, and so $h(b) - h(a) \leqslant -K(b-a)$. This means that $-f(b) - (-f(a)) \leqslant -K(b-a)$. Multiply through by -1 to get $f(b) - f(a) \geqslant K(b-a)$, so the mean value inequality works for $\geqslant$ as well as $\leqslant$.

To get part (a), apply this latter inequality to $g(x)$ with $K = 0$.

(b) The converse says that if $g(b) - g(a) \geqslant 0$ for all $b > a$, then $g'(x) \geqslant 0$ for all x. So let $\varepsilon > 0$ be given, and suppose that $g'(x) < -\varepsilon$ for some x. Then there is a $\delta > 0$ such that, if $0 < h < \delta$ then

$$\frac{g(x+h) - g(x)}{h} - g'(x) < \frac{1}{2}\varepsilon$$

(since we know $g(x+h) \geqslant g(x)$, we don't need the absolute value signs), which implies that

$$g(x+h) - g(x) < -\frac{1}{2}\varepsilon h < 0,$$

which is a contradiction. Therefore $g'(x) \geqslant -\varepsilon$ for every $\varepsilon > 0$, which implies that $g'(x) \geqslant 0$.

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4. Define the function $f\colon [0,1] \to [0,1]$ via

$$f(x) = \begin{cases} x & \text{if} \quad x \in \mathbb{Q} \\ 1 - x & \text{if} \quad x \notin \mathbb{Q} \end{cases}$$

For each of the following statements, prove the ones that are true and show that the others are false.

1. $f(f(x)) = x$ for all $x \in [0,1]$.
2. $f(x) + f(1-x) = 1$ for all $x \in [0,1]$.
3. f is bijective (one-to-one and onto).
4. f is everywhere discontinuous on $[0,1]$.
5. $f(x+y) - f(x) - f(y)$ is rational for all $x, y \in [0,1]$ such that $x + y \in [0,1]$.

1. This is true – since it is true for all rationals because $f(f(x)) = f(x) = x$ for them, and for all irrationals because $f(f(x)) = f(1-x) = 1-(1-x) = x$ for them (since $1-x$ is irrational if x is).

2. This is true – since if x is rational so is $1-x$ and so $f(x) + f(1-x) = x + (1-x) = 1$, and if x is irrational then so is $1-x$, so $f(x) + f(1-x) = (1-x) + x = 1$.

3. This is true – since both functions x and $1-x$ are bijective and map rationals to rationals, or irrationals to irrationals, respectively.

4. This is false! $f(x)$ is continuous at $x = \frac{1}{2}$ (but only there).

5. This is true – If x and y are rational, then $x+y$ is rational, so $f(x+y) - f(x) - f(y) = (x+y) - x - y = 0$ is rational. If x is rational and y is irrational, then $x+y$ is irrational, so $f(x+y) - f(x) - f(y) = (1-x-y) - x - (1-y) = -2x$ is rational. If x is irrational and y is rational, then $x+y$ is irrational, so $f(x+y) - f(x) - f(y) = (1-x-y) - (1-x) - y = -2y$ is rational.

If x and y are both irrational, then $x+y$ could be either rational or irrational. If $x+y$ is rational, then $f(x+y) - f(x) - f(y) = x + y - (1-x) - (1-y) = 2x + 2y - 2$, which is rational. If $x+y$ is irrational, then $f(x+y) - f(x) - f(y) = (1-x-y) - (1-x) - (1-y) = -1$, which is rational.

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5. Suppose $\{x_n\}$ is a bounded sequence that satisfies

$$x_n \leqslant \frac{x_{n-1} + x_{n+1}}{2}$$

for all $n \geqslant 2$. Show that the sequence converges (*Hint*: Perhaps first show that the sequence $x_n - x_{n-1}$ is increasing).

What if instead you have

$$x_n \geqslant \frac{x_{n-1} + x_{n+1}}{2}$$

for all $n \geqslant 2$?

If $x_n \leqslant \frac{1}{2}(x_{n-1} + x_{n+1})$, then $2x_n \leqslant x_{n-1} + x_{n+1}$, or $x_n - x_{n-1} \leqslant x_{n+1} - x_n$, so the sequence of differences is increasing (or at least non-decreasing). But this implies that the sequence x_n is decreasing (non-increasing), since if $x_k - x_{k-1} = c > 0$ for some k, then $x_\ell - x_{\ell-1} \geqslant c$ for every $\ell > k$, and by induction we would have $x_{k+m} > x_k + mc$ for all $m > 0$, which grows without bound

contradicting the hypothesis that $\{x_n\}$ is a bounded sequence. So $\{x_n\}$ is a bounded monotonic sequence and hence is convergent by the fundamental axiom.

On the other hand, if $x_n \geqslant \frac{1}{2}(x_{n-1} + x_{n+1})$, then $2x_n \geqslant x_{n-1} + x_{n+1}$, or $x_n - x_{n-1} \geqslant x_{n+1} - x_n$, so the sequence of differences is decreasing (or at least non-inecreasing). But this implies that the sequence x_n is increasing (non-decreasing), since if $x_k - x_{k-1} = c < 0$ for some k, then $x_\ell - x_{\ell-1} \leqslant c$ for every $\ell > k$, and by induction we would have $x_{k+m} > x_k - mc$ for all $m > 0$, which decreases without bound contradicting the hypothesis that $\{x_n\}$ is a bounded sequence. So $\{x_n\}$ is a bounded monotonic sequence and hence is convergent by the fundamental axiom.
