MATH 360 – Homework 2
Due Monday, September 18, 2017

To discuss in recitation (the first two are from Homework 1):

1. Prove Lemma 4, using the proof of Lemma 2 as a guide.

2. Use Theorem 8 to prove: If \( f: [a, b] \to \mathbb{R} \) is continuous and \( f(a) \geq C \geq f(b) \), then there exists \( c \in [a, b] \) such that \( f(c) = C \).

3. (a) Give an “\( \varepsilon-N \)” definition of what it means for a sequence not to converge. Use your definition to show that \( x_n = (-1)^n \) is not convergent.

   (b) Give an “\( \varepsilon-\delta \)” definition of what it means for a function not to be continuous at \( x = a \). Use your definition to show that 
   \[
   f(x) = \begin{cases} 
   \frac{x}{|x|} & \text{if } x \neq 0 \\
   0 & \text{if } x = 0
   \end{cases}
   \]
   is not continuous at \( x = 0 \).

4. Suppose the function \( f: (0, 1) \to \mathbb{R} \) is continuous and satisfies \( 0 < f(x) < x \). Define the sequence of functions \( f_n: (0, 1) \to \mathbb{R} \) by 
   \[
   f_1(x) = f(x), \quad f_n(x) = f(f_{n-1}(x)) \quad \text{for } n \geq 2.
   \]
   Prove that \( f_n(x) \to 0 \) as \( n \to \infty \) for all \( x \in (0, 1) \).

To be handed in on September 18 (the first three are from Homework 1):

1. Show that any real polynomial of odd degree has at least one root. Is the result true for polynomials of even degree? Proof or counterexample.

   Write the polynomial as \( p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \), where \( a_n \neq 0 \) and \( n \) is odd. Since \( a_n \neq 0 \) we can divide the equation \( p(x) = 0 \) through by \( a_n \) and assume that the polynomial is monic, i.e., that the leading coefficient is 1. In fact we will assume we have done this, so the polynomial we are working with is 
   \[
   m(x) = x^n + \sum_{k=0}^{n-1} b_k x^k.
   \]

   If \( |x| > M = n \max\{ |b_k| + 1, k = 0 \ldots n-1 \} \), then 
   \[
   \left| \sum_{k=0}^{n-1} b_k x^k \right| \leq \sum_{k=0}^{n-1} |b_k| |x|^k < n \max\{ |b_k|, k = 0 \ldots n-1 \} |x|^{n-1} < |x|^n.
   \]
Then for positive \( x > M \), we have \( x^n > 0 \) and in fact \( m(x) > x^n - x^n = 0 \). Likewise for negative \( x < -M \) we have \( x^n < 0 \) (since \( n \) is odd) and \( m(x) < x^n - x^n = 0 \). So we can apply the intermediate value theorem to conclude that there is a \( c \) with \( -M < c < M \) where \( f(c) = 0 \).

2. (a) Suppose that \( g: [0, 1] \rightarrow [0, 1] \) is a continuous function. Show that there exists a \( c \in [0, 1] \) with \( g(c) = c \) (i.e., every continuous map of \( [0, 1] \) to itself has a “fixed point”). (Hint: consider \( f(x) = g(x) - x \))

(b) Give an example of a bijective (one-to-one and onto) function \( h: (0, 1) \rightarrow (0, 1) \) such that \( h(x) \neq x \) for all \( x \in (0, 1) \).

This exercise shows that there is an essential difference between open and closed intervals.

(a) Following the hint, let \( f(x) = g(x) - x \). If \( g(0) = 0 \) then \( c = 0 \) is a fixed point. Otherwise, we have \( g(0) > 0 \), so that \( f(0) > 0 \). Next, if \( g(1) = 1 \) then \( c = 1 \) is a fixed point. Otherwise we have \( g(1) < 1 \), so that \( f(1) < 0 \). Therefore (since \( f \) is continuous, being the sum of continuous functions), there is a point \( c \) in \( [0, 1] \) where \( f(c) = 0 \), and so \( g(c) = c \).

(b) The function \( h(x) = x^2 \) is bijective from \( (0, 1) \) to \( (0, 1) \) with \( h(x) = x^2 \neq x \).

3. (a) Suppose \( g: (A, B) \rightarrow \mathbb{R} \) is a differentiable function such that \( g'(x) \geq 0 \) for all \( x \in (A, B) \). For \( a, b \in (A, B) \) with \( b > a \), show that \( g(b) - g(a) \geq 0 \).

(b) Prove the converse of the result it part (a) (you might want to prove that \( g'(x) \geq -\varepsilon \) for all \( \varepsilon > 0 \)).

(a) The mean value inequality says that if \( f'(x) \leq K \) on \( [a, b] \) then \( f(b) - f(a) \leq K(b - a) \).

Now suppose \( f'(x) \geq K \) on \( [a, b] \). Then the function \( h(x) = -f(x) \) satisfies \( h'(x) \leq -K \), and so \( h(b) - h(a) \leq -K(b - a) \). This means that \( -f(b) - (-f(a)) \leq -K(b - a) \). Multiply through by \(-1\) to get \( f(b) - f(a) \geq K(b - a) \), so the mean value inequality works for \( \geq \) as well as \( \leq \).

To get part (a), apply this latter inequality to \( g(x) \) with \( K = 0 \).

(b) The converse says that if \( g(b) - g(a) \geq 0 \) for all \( b > a \), then \( g'(x) \geq 0 \) for all \( x \). So let \( \varepsilon > 0 \) be given, and suppose that \( g'(x) < -\varepsilon \) for some \( x \). Then there is a \( \delta > 0 \) such that, if \( 0 < h < \delta \) then

\[
\frac{g(x + h) - g(x)}{h} - g'(x) < \frac{1}{2} \varepsilon
\]

(since we know \( g(x + h) \geq g(x) \), we don’t need the absolute value signs), which implies that

\[
g(x + h) - g(x) < \frac{1}{2} \varepsilon h < 0,
\]

which is a contradiction. Therefore \( g'(x) \geq -\varepsilon \) for every \( \varepsilon > 0 \), which implies that \( g'(x) \geq 0 \).

4. Define the function \( f: [0, 1] \rightarrow [0, 1] \) via

\[
f(x) = \begin{cases} 
  x & \text{if } x \in \mathbb{Q} \\
  1 - x & \text{if } x \notin \mathbb{Q}
\end{cases}
\]
For each of the following statements, prove the ones that are true and show that the others are false.

1. $f(f(x)) = x$ for all $x \in [0, 1]$.
2. $f(x) + f(1 - x) = 1$ for all $x \in [0, 1]$.
3. $f$ is bijective (one-to-one and onto).
4. $f$ is everywhere discontinuous on $[0, 1]$.
5. $f(x + y) - f(x) - f(y)$ is rational for all $x, y \in [0, 1]$ such that $x + y \in [0, 1]$.

1. This is true – since it is true for all rationals because $f(f(x)) = f(x) = x$ for them, and for all irrationals because $f(f(x)) = f(1 - x) = 1 - (1 - x) = x$ for them (since $1 - x$ is irrational if $x$ is).

2. This is true – since if $x$ is rational so is $1 - x$ and so $f(x) + f(1 - x) = x + (1 - x) = 1$, and if $x$ is irrational then so is $1 - x$, so $f(x) + f(1 - x) = (1 - x) + x = 1$.

3. This is true – since both functions $x$ and $1 - x$ are bijective and map rationals to rationals, or irrationals to irrationals, respectively.

4. This is false! $f(x)$ is continuous at $x = \frac{1}{2}$ (but only there).

5. This is true – If $x$ and $y$ are rational, then $x + y$ is rational, so $f(x + y) - f(x) - f(y) = (x + y) - x - y = 0$ is rational. If $x$ is rational and $y$ is irrational, then $x + y$ is irrational, so $f(x + y) - f(x) - f(y) = (1 - x - y) - x - (1 - y) = -2x$ is rational. If $x$ is irrational and $y$ is rational, then $x + y$ is irrational, so $f(x + y) - f(x) - f(y) = (1 - x - y) - (1 - x) - y = -2y$ is rational.

If $x$ and $y$ are both irrational, then $x + y$ could be either rational or irrational. If $x + y$ is rational, then $f(x + y) - f(x) - f(y) = x + y - (1 - x) - (1 - y) = 2x + 2y - 2$, which is rational. If $x + y$ is irrational, then $f(x + y) - f(x) - f(y) = (1 - x - y) - (1 - x) - (1 - y) = -1$, which is rational.

5. Suppose $\{x_n\}$ is a bounded sequence that satisfies

$$x_n \leq \frac{x_{n-1} + x_{n+1}}{2}$$

for all $n \geq 2$. Show that the sequence converges (Hint: Perhaps first show that the sequence $x_n - x_{n-1}$ is increasing).

What if instead you have

$$x_n \geq \frac{x_{n-1} + x_{n+1}}{2}$$

for all $n \geq 2$?

If $x_n \leq \frac{1}{2}(x_{n-1} + x_{n+1})$, then $2x_n \leq x_{n-1} + x_{n+1}$, or $x_n - x_{n-1} \leq x_{n+1} - x_n$, so the sequence of differences is increasing (or at least non-decreasing). But this implies that the sequence $x_n$ is decreasing (non-increasing), since if $x_k - x_{k-1} = c > 0$ for some $k$, then $x_{\ell} - x_{\ell-1} \geq c$ for every $\ell > k$, and by induction we would have $x_{k+m} > x_k + mc$ for all $m > 0$, which grows without bound
contradicting the hypothesis that \( \{x_n\} \) is a bounded sequence. So \( \{x_n\} \) is a bounded monotonic sequence and hence is convergent by the fundamental axiom.

On the other hand, if \( x_n \geq \frac{1}{2}(x_{n-1} + x_{n+1}) \), then \( 2x_n \geq x_{n-1} + x_{n+1} \), or \( x_n - x_{n-1} \geq x_{n+1} - x_n \), so the sequence of differences is decreasing (or at least non-increasing). But this implies that the sequence \( x_n \) is increasing (non-decreasing), since if \( x_k - x_{k-1} = c < 0 \) for some \( k \), then \( x_{\ell} - x_{\ell-1} \leq c \) for every \( \ell > k \), and by induction we would have \( x_{k+m} > x_k - mc \) for all \( m > 0 \), which decreases without bound contradicting the hypothesis that \( \{x_n\} \) is a bounded sequence. So \( \{x_n\} \) is a bounded monotonic sequence and hence is convergent by the fundamental axiom.