To discuss in recitation:


5. Textbook page 110, problem 22.

To be handed in on September 25:


   We need to think about the limit
   \[ \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0 - h)}{h} = \lim_{h \to 0} \left( \frac{f(x_0 + h) - f(x_0)}{h} + \frac{f(x_0) - f(x_0 - h)}{h} \right) \]
   The limit of both fractions in the last sum is \( f'(x_0) \), so the limit is \( 2f'(x_0) \).


   If \( f : \mathbb{R} \to \mathbb{R} \) is differentiable and odd, then \( f(-x) = -f(x) \), and by the chain rule we have
   \[ \frac{d}{dx} f(-x) = f'(-x) \frac{d}{dx} (-x) = -f'(-x) \]
   and on the other hand
   \[ \frac{d}{dx} (-f(x)) = -f'(x) \]
   by linearity of the derivative. Therefore \( -f'(-x) = -f'(x) \), or \( f'(-x) = f'(x) \), so \( f' : \mathbb{R} \to \mathbb{R} \) is an even function. Of course, it works the other way around, too.

3. Textbook page 110, problem 23

   We have that \( f(x) \leq 0 \) for all \( x \in \mathbb{R} \) and \( f''(x) \geq 0 \) for all \( x \in \mathbb{R} \). Suppose there is a point \( x_0 \) where \( f'(x) \neq 0 \). If \( f'(x_0) > 0 \) then by the mean-value theorem applied to \( f'(x) \),
   \[ f'(x) \geq f'(x_0) \quad \text{for all} \quad x > x_0. \]
Now consider the linear function \( L(x) = f(x_0) + f'(x_0)(x - x_0) \) (this is the equation of the tangent line to the graph of \( y = f(x) \) at \( x = x_0 \)). We have \( L(x_0) = f(x_0) \) and \( L'(x) \leq f'(x) \) for all \( x > x_0 \). Therefore (by the mean value theorem applied to \( f - L \)),

\[
    f(x) \geq L(x) \quad \text{for all } x > x_0.
\]

But we can solve \( L(x) = 0 \) for \( x \): At

\[
    x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}
\]

we have \( L(x_1) = 0 \) and so for any \( x_2 > x_1 \) we will have \( L(x_2) > 0 \). Therefore (because \( f(x_0) \leq 0 \) and \( f'(x_0) > 0 \) at \( x_2 > x_1 \geq x_0 \) we will have \( f(x_2) \geq L(x_2) > 0 \), contradicting the fact that \( f(x) \leq 0 \) for all \( x \). Therefore we cannot have \( f'(x_0) > 0 \) at any point.

On the other hand, suppose there is a point \( x_0 \) where \( f'(x_0) < 0 \), then by the mean-value theorem applied to \( f'(x) \),

\[
    f'(x) \leq f'(x_0) \quad \text{for all } x < x_0.
\]

Consider the same linear function \( L(x) = f(x_0) + f'(x_0)(x - x_0) \) (the tangent line at \( x_0 \), keeping in mind that \( f'(x_0) \) is negative). We have \( L(x_0) = f(x_0) \) and \( L'(x) = f'(x_0) \geq f'(x) \) for all \( x < x_0 \). Therefore (by the mean value theorem applied to \( f - L \)),

\[
    f(x) \geq L(x) \quad \text{for all } x < x_0
\]

(because \( L \) decreases less than \( f \) does, but they end up in the same place).

4. (a) Suppose that \( f : [a, b] \to \mathbb{R} \) is an increasing (but not necessarily continuous) function and \( g : [a, b] \to \mathbb{R} \) is continuous. If \( f(a) \geq g(a) \) and \( f(b) \leq g(b) \), show that there exists a \( c \in [a, b] \) such that \( f(c) = g(c) \).

(b) Is the result of part (a) true if you replace “\( f(a) \geq g(a) \) and \( f(b) \leq g(b) \)” with “\( f(a) \leq g(a) \) and \( f(b) \geq g(b) \)”? Is the result of part (a) true if you replace “\( g \) continuous” by “\( g \) decreasing”? Is it true if you replace “\( f \)” increasing” with “\( f \) continuous”? In each case, give a proof or a counterexample as appropriate.

(a) If \( f(b) = g(b) \) then we’re done, so assume \( f(b) < g(b) \) and let \( S = \{ x \in [a, b] \mid f(x) < g(x) \} \) and set \( c = \inf S \). The goal is to show that \( f(c) = g(c) \), and to do this we’ll show that \( |f(c) - g(c)| < \varepsilon \) for every \( \varepsilon > 0 \).

So let \( \varepsilon > 0 \) be given. Since \( g \) is continuous, there is a \( \delta > 0 \) so that \( |g(x) - g(c)| < \varepsilon \). By the definition of \( c \), for \( c - \delta < x < c \) we have \( f(x) \geq g(x) \) and \( f \) is increasing, so for any \( x_L \) in that interval we have that if \( x > x_L \) then \( f(x) \geq f(x_L) \geq g(x_L) > g(c) - \varepsilon \). In particular, \( f(c) \geq g(c) - \varepsilon \).

Next, we show that \( c \notin S \) by contradiction: If \( c \in S \), then \( f(c) < g(c) \), say \( g(c) - f(c) = d > 0 \), which would contradict the fact the \( f(c) \geq g(c) - \varepsilon \) for \( \varepsilon = \frac{d}{2} \).

Because \( c \notin S \), for any \( \delta > 0 \) there must be an \( x_R \in S \) such that \( x_R > c \) (of course) and \( x_R - c < \delta \). For the \( \delta \) given above, we conclude that \( f(x_R) < g(c) + \varepsilon \).

But now we have \( x_L < c < x_R \), and since \( f \) is increasing, we conclude that

\[
    g(c) - \varepsilon < f(x_L) \leq f(c) \leq f(x_R) < g(c) + \varepsilon,
\]
and so \(|f(c) - g(c)| < \varepsilon\). Since this is true for every \(\varepsilon > 0\), we have \(f(c) = g(c)\).

We can also do the proof by “lion-hunting”. Construct the increasing sequence \(\{a_n\}\) and the decreasing sequence \(\{b_n\}\) by starting with \(a_0 = a\) and \(b_0 = b\) and repeatedly bisecting the interval and letting \(b_{n+1} = b_n\) and \(a_{n+1}\) be the midpoint \(m\) of \([a_n, b_n]\) if \(f(m) > g(m)\), and \(a_{n+1} = a_n\) and \(b_{n+1} = m\) if \(f(m) < g(m)\) (of course if we encounter a midpoint where \(f(m) = g(m)\), then we declare \(c = m\) and are done). So we have \(b_n - a_n = 2^{-n}(b - a)\) and both sequences bounded and monotonic, so both converge to the same limit \(c\). We need to show that \(f(c) = g(c)\).

This part of the proof is like before, in that we show that \(|f(c) - g(c)| < \varepsilon\) for all \(\varepsilon > 0\). So let \(\varepsilon > 0\) be given. Since \(g\) is continuous, there is a \(\delta > 0\) so that \(|g(x) - g(c)| < \varepsilon\). And since \(a_n \to c\) and \(a_n\) is increasing, for some value of \(n\) we have \(a_n > c - \delta\). Since \(f(a_n) > g(a_n)\) and \(f\) is increasing, we have \(f(c) \geq f(a_n) > g(a_n) > g(c) - \varepsilon\). In particular, \(f(c) \geq g(c) - \varepsilon\). Likewise, since \(b_n \to c\) and \(b_n\) is decreasing, for some value of \(n\) we have \(b_n < c + \delta\). Since \(f(b_n) < g(b_n)\) and \(f\) increasing, we have \(f(c) \geq f(b_n) < g(b_n) < g(c) + \varepsilon\). In particular \(f(c) \leq g(c) + \varepsilon\).

Therefore, \(|f(c) - g(c)| < \varepsilon\). Since this is true for every \(\varepsilon > 0\), we have \(f(c) = g(c)\).

(b) The statement is false if you reverse the inequalities: Let \(g(x) = 0\) for all \(x\) and

\[
f(x) = \begin{cases} 
  x - 3 & \text{if } x < 0 \\
  x + 3 & \text{if } x \geq 0.
\end{cases}
\]

Then \(f(-1) = -4 \leq 0 = g(-1)\) and \(f(+1) = 4 \geq 0 = g(+1)\) and \(f\) is increasing and \(g\) is continuous, but there is no point where \(f(x) = 0\).

I guess the result is true if \(g\) is decreasing, since then we have \(g(a) \geq g(b) \geq f(b) \geq f(a) \geq g(a)\), so all the “\(\geq\)’s” must be equalities, and both functions are the same constant, so \(c\) can be anything.

Of course the result is true if \(f\) is also continuous — this is the intermediate value theorem.

5. Prove that the composition of continuous functions is continuous twice, each time using a different characterization of continuity (e.g., \(\varepsilon\)-\(\delta\), inverse image of open sets, convergent sequences, etc.)

We’re given \(f: V \to W\) and \(g: U \to V\) and we want to show that \(f \circ g: U \to W\) is continuous.

\(\varepsilon\)-\(\delta\): Let \(x \in U\), so \(g(x) \in V\) and \(f(g(x)) \in W\). Given \(\varepsilon\) we know there is an \(\eta\) so that if \(|v - g(x)| < \eta\), then \(|f(v) - f(g(x))| < \varepsilon\) because \(f\) is continuous. And we also know that there is a \(\delta\) so that if \(|u - x| < \delta\) then \(|g(u) - g(x)| < \eta\) because \(g\) is continuous. So for this \(\delta\), if we have \(|u - x| < \delta\) then \(|f(g(u)) - f(g(x))| < \varepsilon\). Not so bad.

Open sets: Let \(G\) be an open set in \(W\). Because \(f\) is continuous, \(f^{-1}(G)\) is open in \(V\). And then because \(g\) is continuous, \(g^{-1}(f^{-1}(G))\) is open in \(U\). But \(g^{-1}(f^{-1}(G)) = (f \circ g)^{-1}(G)\), so we are done.

Sequences: Let \(x_n \to x\) in \(U\). Because \(g\) is continuous, \(g(x_n) \to g(x)\) in \(V\). And because \(f\) is continuous, \(f(g(x_n)) \to f(g(x))\) in \(W\), and once again, we are done.