MATH 360 –Homework 3

Due Monday, September 25, 2017

To discuss in recitation:

1. Textbook page 62, problem 7.

---

2. Textbook page 86, problem 11.

---

3. Textbook page 95, problem 13.

---

4. Textbook page 101, problem 5.

---

5. Textbook page 110, problem 22.

To be handed in on September 25:

1. Textbook page 95, problem 14.

We need to think about the limit

$$\lim_{h\to 0} \frac{f(x_0+h)-f(x_0-h)}{h} = \lim_{h\to 0}\left(\frac{f(x_0+h)-f(x_0)}{h} + \frac{f(x_0)-f(x_0-h)}{h}\right)$$

The limit of both fractions in the last sum is $f'(x_0)$, so the limit is $2f'(x_0)$.

---

2. Textbook page 101, problem 9.

If $f\colon \mathbb{R}\to\mathbb{R}$ is differentiable and odd, then $f(-x) = -f(x)$, and by the chain rule we have

$$\frac{d}{dx}f(-x) = f'(-x)\frac{d}{dx}(-x) = -f'(-x)$$

and on the other hand

$$\frac{d}{dx}(-f(x)) = -f'(x)$$

by linearity of the derivative. Therefore $-f'(-x) = -f'(x)$, or $f'(-x) = f'(x)$, so $f'\colon \mathbb{R}\to\mathbb{R}$ is an even function. Of course, it works the other way around, too.

---

3. Textbook page 110, problem 23

We have that $f(x) \leqslant 0$ for all $x \in \mathbb{R}$ and $f''(x) \geqslant 0$ for all $x \in \mathbb{R}$. Suppose there is a point x_0 where $f'(x) \neq 0$. If $f'(x_0) > 0$ then by the mean-value theorem applied to $f'(x)$,

$$f'(x) \geqslant f'(x_0) \quad \text{for all } x > x_0.$$

Now consider the linear function $L(x) = f(x_0) + f'(x_0)(x - x_0)$ (this is the equation of the tangent line to the graph of $y = f(x)$ at $x = x_0$). We have $L(x_0) = f(x_0)$ and $L'(x) \leqslant f'(x)$ for all $x > x_0$. Therefore (by the mean value theorem applied to $f - L$),

$$f(x) \geqslant L(x) \quad \text{for all } x > x_0.$$

But we can solve $L(x) = 0$ for x: At

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

we have $L(x_1) = 0$ and so for any $x_2 > x_1$ we will have $L(x_2) > 0$). Therefore (because $f(x_0) \leqslant 0$ and $f'(x_0) > 0$) at $x_2 > x_1 \geqslant x_0$ we will have $f(x_2) \geqslant L(x_2) > 0$, contradicting the fact that $f(x) \leqslant 0$ for all x. Therefore we cannot have $f'(x_0) > 0$ at any point.

On the other hand, suppose there is a point x_0 where $f'(x_0) < 0$, then by the mean-value theorem applied to $f'(x)$,

$$f'(x) \leqslant f'(x_0) \quad \text{for all } x < x_0.$$

Consider the same linear function $L(x) = f(x_0) + f'(x_0)(x - x_0)$ (the tangent line at x_0, keeping in mind that $f'(x_0)$ is *negative*). We have $L(x_0) = f(x_0)$ and $L'(x) = f'(x_0) \geqslant f'(x)$ for all $x < x_0$. Therefore (by the mean value theorem applied to $f - L$),

$$f(x) \geqslant L(x) \quad \text{for all } x < x_0$$

(because L decreases less than f does, but they end up in the same place).

---

4. (a) Suppose that $f\colon [a, b] \to \mathbb{R}$ is an increasing (but not necessarily continuous) function and $g\colon [a, b] \to \mathbb{R}$ is continuous. If $f(a) \geqslant g(a)$ and $f(b) \leqslant g(b)$, show that there exists a $c \in [a, b]$ such that $f(c) = g(c)$.

(b) Is the result of part (a) true if you replace "$f(a) \geqslant g(a)$ and $f(b) \leqslant g(b)$" with "$f(a) \leqslant g(a)$ and $f(b) \geqslant g(b)$"? Is the result of part (a) true if you replace "g continuous" by "g decreasing"? Is it true if you replace "f increasing" with "f continuous"? In each case, give a proof or a counterexample as appropriate.

(a) If $f(b) = g(b)$ then we're done, so assume $f(b) < g(b)$ and let $S = \{x \in [a, b] \mid f(x) < g(x)\}$ and set $c = \inf S$. The goal is to show that $f(c) = g(c)$, and to do this we'll show that $|f(c) - g(c)| < \varepsilon$ for every $\varepsilon > 0$.

So let $\varepsilon > 0$ be given. Since g is continuous, there is a $\delta > 0$ so that $|g(x) - g(c)| < \varepsilon$. By the definition of c, for $c - \delta < x < c$ we have $f(x) \geqslant g(x)$ and f is increasing, so for any x_L in that interval we have that if $x > x_L$ then $f(x) \geqslant f(x_L) \geqslant g(x_L) > g(c) - \varepsilon$. In particular, $f(c) \geqslant g(c) - \varepsilon$.

Next, we show that $c \notin S$ by contradiction: If $c \in S$, then $f(c) < g(c)$, say $g(c) - f(c) = d > 0$, which would contradict the fact the $f(c) \geqslant g(c) - \varepsilon$ for $\varepsilon = \frac{d}{2}$.

Because $c \notin S$, for any $\delta > 0$ there must be an $x_R \in S$ such that $x_R > c$ (of course) and $x_R - c < \delta$. For the δ given above, we conclude that $f(x_R) < g(c) + \varepsilon$.

But now we have $x_L < c < x_R$, and since f is increasing, we conclude that

$$g(c) - \varepsilon < f(x_L) \leqslant f(c) \leqslant f(x_r) < g(c) + \varepsilon,$$

and so $|f(c) - g(c)| < \varepsilon$. Since this is true for every $\varepsilon > 0$, we have $f(c) = g(c)$.

We can also do the proof by "lion-hunting". Construct the increasing sequence $\{a_n\}$ and the decreasing sequence $\{b_n\}$ by starting with $a_0 = a$ and $b_0 = b$ and repeatedly bisecting the interval and letting $b_{n+1} = b_n$ and a_{n+1} be the midpoint m of $[a_n, b_n]$ if $f(m) > g(m)$, and $a_{n+1} = a_n$ and $b_{n+1} = m$ if $f(m) < g(m)$ (of course if we encounter a midpoint where $f(m) = g(m)$, then we declare $c = m$ and are done). So we have $b_n - a_n = 2^{-n}(b - a)$ and both sequences bounded and monotonic, so both converge to the same limit c. We need to show that $f(c) = g(c)$.

This part of the proof is like before, in that we show that $|f(c) - g(c)| < \varepsilon$ for all $\varepsilon > 0$. So let $\varepsilon > 0$ be given. Since g is continuous, there is a $\delta > 0$ so that $|g(x) - g(c)| < \varepsilon$. And since $a_n \to c$ and a_n is increasing, for some value of n we have $a_n > c - \delta$. Since $f(a_n) > g(a_n)$ and f is increasing, we have $f(c) \geqslant f(a_n) > g(a_n) > g(c) - \varepsilon$. In particular, $f(c) \geqslant g(c) - \varepsilon$.

Likewise, since $b_n \to c$ and b_n is decreasing, for some value of n we have $b_n < c + \delta$. Since $f(b_n) < g(b_n)$ and f increasing, we have $f(c) \geqslant f(b_n) < g(b_n) < g(c) + \varepsilon$. In particular $f(c) \leqslant g(c) + \varepsilon$.

Therefore, $|f(c) - g(c)| < \varepsilon$. Since this is true for every $\varepsilon > 0$, we have $f(c) = g(c)$.

(b) The statement is false if you reverse the inequalities: Let $g(x) = 0$ for all x and

$$f(x) = \begin{cases} x - 3 & \text{if} \quad x < 0 \\ x + 3 & \text{if} \quad x \geqslant 0. \end{cases}$$

Then $f(-1) = -4 \leqslant 0 = g(-1)$ and $f(+1) = 4 \geqslant 0 = g(+1)$ and f is increasing and g is continuous, but there is no point where $f(x) = 0$.

I guess the result is true if g is decreasing, since then we have $g(a) \geqslant g(b) \geqslant f(b) \geqslant f(a) \geqslant g(a)$, so all the "$\geqslant$"s must be equalities, and both functions are the same constant, so c can be anything.

Of course the result is true if f is also continuous — this is the intermediate value theorem.

---

5. Prove that the composition of continuous functions is continuous twice, each time using a different characterization of continuity (e.g., ε-δ, inverse image of open sets, convergent sequences, etc.)

We're given $f\colon V \to W$ and $g\colon U \to V$ and we want to show that $f \circ g\colon U \to W$ is continuous.

ε-δ: Let $x \in U$, so $g(x) \in V$ and $f(g(x)) \in W$. Given ε we know there is an η so that if $|v - g(x)| < \eta$, then $|f(v) - f(g(x))| < \varepsilon$ because f is continuous. And we also know that there is a δ so that if $|u - x| < \delta$ then $|g(u) - g(x)| < \eta$ because g is continuous. So for this δ, if we have $|u - x| < \delta$ then $|f(g(u)) - f(g(x))| < \varepsilon$. Not so bad.

Open sets: Let G be an open set in W. Because f is continuous, $f^{-1}(G)$ is open in V. And then because g is continuous, $g^{-1}(f^{-1}(G))$ is open in U. But $g^{-1}(f^{-1}(G)) = (f \circ g)^{-1}(G)$, so we are done.

Sequences: Let $x_n \to x$ in U. Because g is continuous, $g(x_n) \to g(x)$ in V. And because f is continuous, $f(g(x_n)) \to f(g(x))$ in W, and once again, we are done.
