MATH 360 –Homework 6

Due Wednesday, October 25, 2017

To discuss in recitation:

1. Textbook page 173, problems 3, 4

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2. Textbook page 178, problems 3,4.

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3. Textbook page 188, problems 1,3.

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4. Prove the integration by parts formula, and work a few examples.

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5. Textbook page 69, problems 4, 5, 6, 7

To be handed in on October 25:

1. Textbook page 173, problems 5, 6.

Problem 5: First rewrite $G(x)$ as

$$G(x) = x\int_0^x f(t)\,dt - \int_0^x tf(t)\,dt.$$

Now use the product rule and the fundamental theorem to calculate:

$$G'(x) = \int_0^x f(t)\,dt + xf(x) - xf(x) = \int_0^x f(t)\,dt.$$

Therefore $G''(x)$ exists and by the fundamental theorem it is $f(x)$.

Problem 6: Certainly $F(1) = 0$, and since $1/(2\sqrt{t}-1) > 1/(2\sqrt{t}) > 0$, we have that $F'(x) > 0$ on $[1,\infty)$ so $F(a) < F(b)$ whenever $a < b$, so there can be at most one solution of $F(x) = c$.

$$F(x)\int_1^x \frac{1}{2\sqrt{t}-1}\,dt > \int_1^x \frac{1}{2\sqrt{t}}\,dt = \sqrt{x} - 1.$$

So if we choose $b > (c+1)^2$ we will have $F(b) > c$, so by the intermediate value theorem there is a solution of $F(x) = c$ in $[1, b]$.

2. Textbook page 177, problems 2, 5.

Problem 2: If $F(x) = a(1-e^{-cx})$ then $F'(x) = ace^{-cx}$ and $c(a-F(x)) = ac-[ac-ace^{-cx}] = ace^{-cx}$, and $F(0) = a(1-e^0) = 0$, so $F(x)$ is a solution of the problem. Now suppose there were another solution $G(x)$. Then $H(x) = F(x)-G(x)$ would satisfy the differential equation $H'(x) = -cH(x)$ and $H(0) = 0$. But we already knew that the unique solution of this problem is $H(x) = 0$, and therefore $G(x)$ was the same as $F(x)$ after all.

Problem 5: Since f is continuous, the integral on the right side defines a differentiable function (with derivative $f(x)$). Therefore f (the left side) is differentiable and moreover $f'(x) = f(x)$. But $f(0) = 0$ because the integral from 0 to 0 of an integrable function is zero. And we know that the unique solution of this initial-value problem is $f(x) = 0$ for all x.

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3. Textbook page 183, problem 9.

In the integral

$$\int_0^x f(t)\,dt$$

let $s = f(t)$, so $t = f^{-1}(s)$. Then

$$\begin{aligned}\int_0^x f(t)\,dt &= \int_0^{f(x)} f\Big(f^{-1}(s)\Big)(f^{-1})'(s)\,ds = \int_0^{f(x)} s(f^{-1})'(s)\,ds \\ &= sf^{-1}(s)\Big|_0^{f(x)} - \int_0^{f(x)} f^{-1}(s)\,ds \\ &= xf(x) - \int_0^{f(x)} f^{-1}(s)\,ds\end{aligned}$$

(where we integrated by parts with $u = s$ and $dv = (f^{-1})'(s)\,ds$). This proves the formula in problem 8.

(a) If $b = f(a)$ then this is the formula from problem 8 (with $a = x$ and $b = f(x)$ and we have equality. If $b < f(a)$ then let $c = f^{-1}(b) < a$. Then from problem 8,

$$cb = \int_0^c f(s)\,ds + \int_0^b f^{-1}(t)\,dt$$

But then

$$ab = bc + b(a-c) = bc + \int_c^a b\,ds < cb + \int_c^a f(s)\,ds$$

because $f(c) = b$ and f is strictly increasing. But then

$$ab < \int_0^c f(s)\,ds + \int_a^c f(s)\,ds + \int_0^b f^{-1}(t)\,dt = \int_0^a f(s)\,dt + \int_0^b f^{-1}(t)\,dt$$

This proves the result if $b < f(a)$. Interchanging f and f^{-1} gives the result if $b > f(a)$.

(b) If $p > 1$, then $f(x) = x^{p-1}$ satisfies the hypotheses of Young's Inequality ($f(0) = 0$ and f is strictly increasing and goes to ∞ as $x \to \infty$). Now $f^{-1}(x) = x^{1/(p-1)}$, and if we apply the result of (a) with $f(x) = x^{p-1}$ and $f^{-1}(x) = x^{1/(p-1)}$ (and note that $1 + 1/(p-1) = p/(p-1)$) we get

$$ab \leqslant \int_0^a x^{p-1}\,dx + \int_0^b x^{1/(p-1)}\,dx = \frac{a^p}{p} + \frac{b^{p/p-1}}{p/(p-1)} = \frac{a^p}{p} + \frac{b^q}{q}$$

where $q = p/(p-1)$. Finally, note that

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{p} + \frac{p-1}{p} = 1.$$

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4. Suppose f is a continuously differentiable function on $[a, b]$ and that $f(a) = f(b) = 0$. Further, assume that

$$\int_a^b (f(x))^2\,dx = 1.$$

(a) Prove that

$$\int_a^b x f(x) f'(x)\,dx = -\frac{1}{2}$$

(b) Also prove that

$$\int_a^b (f'(x))^2\,dx \cdot \int_a^b x^2 (f(x))^2\,dx \geqslant \frac{1}{4}.$$

(c) For what function(s) (if any) is equality achieved in (b)?

(a) Integrating by parts, with $u = (f(x))^2$ and $dv = dx$ we have $du = 2f(x)f'(x)$ and $v = x$, so

$$1 = \int_a^b (f(x))^2\,dx = x(f(x))^2\Big|_a^b - \int_a^b 2xf(x)f'(x)\,dx.$$

The first term is zero because $f(a) = f(b) = 0$ and so

$$1 = -\int_a^b x f(x) f'(x)\,dx.$$

Dividing by $-1/2$ gives the result.

(b) This follows from part (a) by applying the Cauchy-Schwarz inequality to the pair of functions $f'(x)$ and $xf(x)$:

$$\frac{1}{4} = \left(\int_a^b [xf(x)]\,[f'(x)]\,dx \right)^2 \leqslant \int_a^b [xf(x)]^2\,dx \cdot \int_a^b [f'(x)]^2\,dx.$$

(c) you get equality in Cauchy-Schwarz when the two functions are constant multiples of one another. So we need to check whether there are any functions $f(x)$ on $[a,b]$ with $f(a) = f(b) = 0$ and with the integral of $(f(x))^2$ being 1, for which $f'(x) = Kxf(x)$. Divide this (differential) equation by $f(x)$ and note that the left side becomes the derivative of $\ln(f(x)$. Now integrate both sides and get $\ln(f(x)) = \frac{1}{2}Kx^2 + C$, so $f(x) = \tilde{C}e^{Kx^2}$ (where $\tilde{C} = e^C$). But such a function cannot be zero anywhere unless it is identically zero, but then the integral of its square wouldn't be 1. So there are no functions under consideration for which equality holds.

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5. Textbook page 69, problems 9, 10, 11

Problem 9: If D were an interval then the answer would be yes. But if D is just a set, then it could be a discrete set (say the set $\mathbb{Z}$) in which case any function is uniformly continuous on D, since for δ small enough there's only one point y such that $|x-y| < \delta$, namely x itself.

Problem 10: Let $m = \frac{1}{2}(a+b)$ be the midpoint of the interval. Choose $\varepsilon = 1$, and let $\delta > 0$ be the number guaranteed by the uniform continuity so that for all $x, y \in (a,b)$ with $|x-y| < \delta$, we have $|f(x) - f(y)| < 1$. Since $\frac{1}{2}\delta < \delta$ we know that $|f(x) - f(m)| < 1$ for all $m - \frac{1}{2}\delta \leqslant x \leqslant m + \frac{1}{2}\delta$. But then $|f(x) - f(m)| < 2$ for all $m - \frac{2}{2}\delta \leqslant x \leqslant m + \frac{2}{2}\delta$. Keep doing this and getting $|f(x) - f(m)| < k$ for all $m - \frac{k}{2}\delta \leqslant x \leqslant m + \frac{k}{2}\delta$. There must be a k_1 such that $m + \frac{k_1}{2}\delta > b$ and k_2 so that $m - \frac{k_1}{2}\delta < a$, and the process can stop, and we'll have that $|f(x)| < |f(m)| + \max\{k_1, k_2\}$ for all $x \in (a,b)$.

Problem 11: Given ε, we can let $\delta = \varepsilon/C$. Then for x, y such that $|x-y| < \delta$ we'll have

$$|f(x) - f(y)| < C|x-y| < C \cdot \frac{\varepsilon}{C} = \varepsilon,$$

so f is uniformly continuous.