1. Let $f: [a, b] \to \mathbb{R}$ be a continuous function, and suppose that for every continuous function $g: [a, b] \to \mathbb{R}$,

$$\int_a^b f(x)g(x) \, dx = 0.$$ 

Prove that $f(x) = 0$ for all $x \in [a, b]$. 

2. Let $v(t)$ be the velocity of an object for $t \in [0, T]$ You know that $v(t)$ is the derivative with respect to $t$ of $x(t)$, the position of the object at time $t$. Assume that $v(t) > 0$ for all $t$.

(a) What is the average velocity with respect to $t$?

(b) Show that $x(t)$ is an invertible function of $t$. What are its domain and range?

(c) Since $x$ is an invertible function of $t$, we can consider $v$ as a function of $x$. Show that the average of $v$ with respect to $x$ is greater than or equal to the average of $v$ with respect to $t$. At a key point, you will need to use one of those famous inequalities.
3. Let $f(x)$ and $g(x)$ be functions for which their Taylor series (centered at $x = 0$) converge to the functions themselves for $x \in [-r, r]$.

(a) Show that the Taylor series for $cf(x)$ converges to $cf(x)$ for $x \in [-r, r]$.

(b) Show that the Taylor series for $xf(x)$ converges to $xf(x)$ for $x \in [-r, r]$.

(c) Show that the Taylor series for $f(x) + g(x)$ converges to $f(x) + g(x)$ for $x \in [-r, r]$.

(d) Suppose the Taylor series for $\varphi(x)$ is

$$f_0 + f_1 x + f_2 x^2 + \cdots$$

where $f_0 = 0$, $f_1 = 1$ and $f_{n+2} = f_{n+1} + f_n$, so $f_n$ is the $n$th Fibonacci number. What is $\varphi(x)$?
4. Let

\[ f(x) = \begin{cases} 
  x & \text{for } 0 \leq x \leq 1 \\
  3 - x & \text{for } 1 < x \leq 2 
\end{cases} \]

Prove carefully (upper and lower sums etc., and “mind the gap”) that \( f \) is integrable on \([0, 2]\) and evaluate the integral.
5. Let

\[ f(x) = \int_0^x \frac{1}{1 + t^4} \, dt. \]

Prove that \( f \) is uniformly continuous on all of \( \mathbb{R} \). **Hint**: Do not attempt to evaluate the integral!!
6. Suppose $g: \mathbb{R} \to \mathbb{R}$, with bounded derivative (say $|g'| \leq M$). Fix $\varepsilon > 0$ and define

$$f(x) = x + \varepsilon g(x).$$

Prove that $f$ is one-to-one if $\varepsilon$ is sufficiently small.
Prove that for every positive integer $n$,

\[
\frac{1}{n+1} < \ln \left(1 + \frac{1}{n}\right) < \frac{1}{n}.
\]