MATH 360 – Final Exam Sample Problems

Thursday, December 14, 2017

1. Let $\{a_n\}$ be a sequence of real numbers such that $\lim_{n\to\infty} a_n = 0$. Prove that the series $\sum_{n=0}^{\infty} a_n x^n$ converges uniformly on the closed interval $-\frac{1}{2} \leqslant x \leqslant \frac{1}{2}$.

Since $\mathbf{a}_n \to 0$, we have $|a_n|$ is bounded, say $|a_n| < M$. Therefore

$$\left|\sum_{n=N}^{\infty} a_n x^n\right| \leqslant \sum_{n=N}^{\infty} |a_n| \left(\frac{1}{2}\right)^n \leqslant M \frac{(\frac{1}{2})^N}{1-\frac{1}{2}} = \frac{M}{2^{N-1}}$$

independent of $x \in [-\frac{1}{2}, \frac{1}{2}]$. So the difference between the Nth partial sum of the series and the sum of the whole series is bounded (and going to zero) independent of x, thus uniformly.

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2. Let $f(x)$ be a continuous function on the closed interval $0 \leqslant x \leqslant 1$, and such that $f(0) = 1$, $f(\frac{1}{2}) = 2$ and $f(1) = 3$. Show that

$$\lim_{n\to\infty} \int_0^1 f(x^n)\, dx$$

exists, and compute the limit.

We claim that $\lim_{n\to\infty} \int_0^1 f(x^n)\, dx = f(0) = 1$. To see this, let $\varepsilon > 0$ be given, and we'll show that there is an $N > 0$ such that for $n > N$ we have

$$\left| f(0 - \int_0^1 f(x^n)\, dx \right| < \varepsilon.$$

First, use the continuity of f on the closed interval $[0,1]$ to assert that f is bounded, say $|f(x)| < M$ for all $x \in [0,1]$. Next, there is a $\delta > 0$ such that $|f(x) - f(0)| < \frac{1}{2}\varepsilon$ provided $0 \leqslant x < \delta$. Finally, there is a $\sigma > 0$ such that $2M\sigma < \frac{1}{2}\varepsilon$, and an N such that $(1-\sigma)^n < \delta$ for all $n > N$. So for $n > N$,

$$\begin{aligned}
\left| f(0) - \int_0^1 f(x^n)\, dx \right| &= \left| \int_0^1 f(0) - f(x^n)\, dx \right| \\
&\leqslant \int_0^{1-\sigma} |f(0) - f(x^n)|\, dx + \int_{1-\sigma}^1 |f(0) - f(x^n)|\, dx \\
&\leqslant \int_0^{1-\sigma} \frac{\varepsilon}{2}\, dx + \int_{1-\sigma}^1 2M\, dx < \int_0^1 \frac{\varepsilon}{2}\, dx + 2M\sigma < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\end{aligned}$$

3. Let $\{b_n\}$ be a sequence of real numbers such that

$$\sum_{n=1}^{\infty} |b_n| = 1$$

and let $f(x)$ be the function given by

$$\sum_{n=1}^{\infty} b_n \cos(nx).$$

Prove that the series converges and that f is continuous on all of $\mathbb{R}$. Is f uniformly continuous?

Since $|\cos(nx)| \leqslant 1$ for all x, we have that the series for $f(x)$ converges uniformly on all of $\mathbb{R}$ by the Weierstrass M-test. Moreover, since f is periodic with period 2π, we get uniform continuity for f from the fact that it is continuous on any closed interval of length greater than 2π.

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4. Let $f\colon X \to Y$ be a continuous mapping from the metric space (X, d_X) to the metric space (Y_d, Y). For $A \subset X$ be a subset of X we write $f(A)$ for $\{y \in Y \mid y = f(a) \text{ for some } a \in A\}$, and for $B \subset Y$ a subset of Y we write $f^{-1}(B)$ for $\{x \in X \mid f(x) \in B\}$. For each of the following, give a proof or a counterexample:

(a) If $A \subset X$ is connected, then $f(A) \subset Y$ is also connected.

(b) If $B \subset Y$ is connected, then $f^{-1}(B) \subset X$ is also connected.

(c) If $A \subset X$ is sequentially compact, then $f(A) \subset Y$ is also sequentially compact.

(d) If $B \subset Y$ is sequentially compact, then $f^{-1}(B) \subset X$ is also sequentially compact.

(a) This is true (and a generalization of the intermediate-value theorem). If P and Q were disjoint open subsets of Y such that $(P \cap f(A)) \cup (Q \cap f(A)) = f(A)$, then their inverse images would be disjoint open sets in X such that $A \cap \left(f^{-1}(P) \cup f^{-1}(Q)\right) = A$ so that A would not be connected either.

(b) This is false. Consider $f(x) = x^2$ on $\mathbb{R}$, and the set $B = [1, \infty)$. We have $f^{-1}(B) = (-\infty, -1] \cup [1, \infty)$ which is not connected.

(c) True. Let $\{y_n\}$ be a sequence in $f(A)$, then there is a sequence $\{x_n\}$ in A with $f(x_n) = y_n$ for each n. Since A is sequentially compact, there is a subsequence of the x_n's that converges (say to x). But then f of that subsequence will converge in $f(A)$, since If $x_n \to x$ in A, then $f(x_n) \to f(x) \in f(A)$ by continuity. So $\{y_n\}$ has a convergent subsequence, and so $f(A)$ is sequentially compact as well.

(d) This is false. Consider $f(x)$ is the constant 0. Then $f^{-1}(\{0\}) = \mathbb{R}$ which is not sequentially compact.

5. Consider the set $\mathbb{Q}$ of rational numbers with its usual metric.

(a) Is every closed, bounded subset of $\mathbb{Q}$ sequentially compact?

(b) Show that every continuous function $f\colon \mathbb{R} \to \mathbb{Q}$ is constant.

Justify your assertions.

(a) No – the set of rational numbers satisfying $0 \leqslant r \leqslant \sqrt{2}$ is closed in $\mathbb{Q}$, but there is a sequence of rationals $\{r_n\}$ in the interval such that $r_n \to \sqrt{2}$. So this is a Cauchy sequence in $\mathbb{Q} \cap [0, \sqrt{2}]$ without a limit.

(b) Since $\mathbb{R}$ is connected, we would have $f(\mathbb{R}) \subset \mathbb{Q}$ is connected. But the only connected subsets of $\mathbb{Q}$ are isolated points, since for any $r_1 < r_2$ in $\mathbb{Q}$, there is an irrational number x such that $r_1 < x < r_2$ and the open sets $(-\infty, x) \cap \mathbb{Q}$ and $(x, \infty) \cap \mathbb{Q}$ are a decomposition of $\mathbb{Q}$ into two disjoint open sets (whose inverse images would be such a decomposition of $\mathbb{R}$).

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6. Suppose f is a twice-differentiable function which satisfies the differential equation

$$\frac{d^2 f}{dx^2} = -(2 + e^{-x}) f(x)^2$$

for $x \geqslant 0$. Suppose $f(0) = 1$ and $f'(0) = 0$. (Do not attempt to solve the equation.) Sketch the graph of f and show that $f(x) = 0$ for one and only one positive value of x.

Certainly $f''(x) \leqslant 0$ for all $x > 0$, and $f''(0) < 0$. Therefore $f'(x) < 0$ for all $x > 0$ by the mean-value theorem. (So the graph sketch would look something like the right half of the parabola $y = 1 - x^2$). Moreover, the graph of f lies below its tangent lines, which have negative slope, so there must be an $x > 0$ where $f(x) = 0$. There can't be two of them or else we'd have $f'(c) \geqslant 0$ for some c between the two solutions.

7. Suppose f is a continuous function defined on the whole real line which is periodic with period one (so $f(x+1) = f(x)$ for all real x). Suppose

$$\int_0^1 f(x)\,dx = 1 \qquad \text{and} \qquad f(0) = 2.$$

Compute the limits

$$\lim_{c\to\infty} \int_0^1 f(cx)\,dx \qquad \text{and} \qquad \lim_{c\to 0} \int_0^1 f(cx)\,dx$$

and justify your answers.

First, make the change of variables $u = cx$ (or $x = u/c$) so $dx = du/c$ and

$$\int_0^1 f(cx)\,dx = \frac{1}{c}\int_0^c f(u)\,du.$$

The limit of this as $c \to 0$ is $f(0)$ by L'Hospital's rule and the fundamental theorem of calculus (among other reasons).

For the limit as $c \to \infty$, use the fact that f is periodic to conclude first that f is bounded, say $|f(x)| < M$ for all $x \in \mathbb{R}$ and that the integral of f over any interval of length 1 is 1. In particular, if $n \leqslant c < n+1$, then (using that same change of variables)

$$\left|\int_0^c f(u)\,du - n\right| = \left|\int_0^n f(u)\,du + \int_n^c f(u)\,du - n\right| = \left|\int_n^c f(u)\,du\right| \leqslant \int_n^{n+1} M\,du = M.$$

Therefore (since $c - 1 < n$ and $c + 1 > n$),

$$\frac{c-1-M}{c} < \frac{1}{c}\int_0^c f(u)\,du < \frac{c+1+M}{c}$$

The limits of both expressions on the ends is 1 as $c \to \infty$, so the middle one approaches 1 as well, which allows us to conclude that

$$\lim_{c\to\infty} \int_0^1 f(cx)\,dx = 1.$$

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8. Say $a_n > 0$ are a sequence of positive real numbers and $a_n \to A$ as $n \to \infty$. Either prove that $A \geqslant 0$ or provide a counterexample.

If A were negative, there would be an N such that $|a_n - A| < \frac{1}{2}|A|$ for all $n > N$. But then

$$a_n < A + \frac{|A|}{2} = \frac{A}{2} < 0,$$

a contradiction of the fact that $a_n > 0$ for all n. Therefore A cannot be negative.

9. For each of the following, give either a proof or a counterexample.

(a) Let f be a continuous real-valued function on the open interval $0 < x < 3$. Must f be uniformly continuous on the open interval $1 < x < 2$?

(b) Suppose instead that f is only assumed to be continuous on the open interval $0 < x < 2$. Must f be uniformly continuous on the open interval $1 < x < 2$?

(a) Yes. Since f is continuous on $(0, 3)$ it is continuous on $[1, 2]$. And a continuous function on a closed, bounded interval is uniformly continuous on that interval. So f is uniformly continuous on $[1, 2]$ and thus on $(1, 2)$.

(b) No. The function $f(x) = \dfrac{1}{2-x}$ is a counterexample (a uniformly continuous function would be bounded).

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10. Write the equivalent integral equation formulation for the initial-value problem $y' = y$, $y(0) = 1$. Then carry out the first few iterations of the contraction mapping proof of existence and show explicitly that they converge to the solution of the initial-value problem.

The integral formulation is

$$y(x) = 1 + \int_0^x y(t)\,dt.$$

So if $y_0(x) \equiv 1$, then

$$y_1(x) = 1 + \int_0^x 1\,dt = 1 + x$$

$$y_2(x) = 1 + \int_0^x (1+t)\,dt = 1 + x + \frac{x^2}{2}$$

$$y_3(x) = 1 + \int_0^x \left(1 + t + \frac{t^2}{2}\right)\,dt = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$$

and you can see that (and by induction)

$$y_n = \sum_{k=0}^{n} = \frac{x^k}{k!}$$

which is the nth partial sum of the series for e^x. This series converges uniformly to e^x on any interval $|x| \leqslant M$.