1. Let \( \{a_n\} \) be a sequence of real numbers such that \( \lim_{n \to \infty} a_n = 0 \). Prove that the series \( \sum_{n=0}^{\infty} a_n x^n \) converges uniformly on the closed interval \(-\frac{1}{2} \leq x \leq \frac{1}{2}\).

Since \( a_n \to 0 \), we have \( |a_n| \) is bounded, say \( |a_n| < M \). Therefore

\[
\left| \sum_{n=N}^{\infty} a_n x^n \right| \leq \sum_{n=N}^{\infty} |a_n| \left( \frac{1}{2} \right)^n \leq M \left( \frac{1}{2} \right)^N \left( \frac{1}{1 - \frac{1}{2}} \right) = \frac{M}{2^{N-1}}
\]

independent of \( x \in \left[ -\frac{1}{2}, \frac{1}{2} \right] \). So the difference between the \( N \)th partial sum of the series and the sum of the whole series is bounded (and going to zero) independent of \( x \), thus uniformly.

2. Let \( f(x) \) be a continuous function on the closed interval \( 0 \leq x \leq 1 \), and such that \( f(0) = 1 \), \( f(\frac{1}{2}) = 2 \) and \( f(1) = 3 \). Show that

\[
\lim_{n \to \infty} \int_{0}^{1} f(x^n) \, dx
\]

exists, and compute the limit.

We claim that \( \lim_{n \to \infty} \int_{0}^{1} f(x^n) \, dx = f(0) = 1 \). To see this, let \( \varepsilon > 0 \) be given, and we’ll show that there is an \( N > 0 \) such that for \( n > N \) we have

\[
\left| f(0) - \int_{0}^{1} f(x^n) \, dx \right| < \varepsilon.
\]

First, use the continuity of \( f \) on the closed interval \([0, 1]\) to assert that \( f \) is bounded, say \( |f(x)| < M \) for all \( x \in [0, 1] \). Next, there is a \( \delta > 0 \) such that \( |f(x) - f(0)| < \frac{1}{2} \varepsilon \) provided \( 0 \leq x < \delta \). Finally, there is a \( \sigma > 0 \) such that \( 2M\sigma < \frac{1}{2} \varepsilon \), and an \( N \) such that \( (1 - \sigma)^n < \delta \) for all \( n > N \). So for \( n > N \),

\[
\begin{align*}
\left| f(0) - \int_{0}^{1} f(x^n) \, dx \right| &= \left| \int_{0}^{1} f(0) - f(x^n) \, dx \right| \\
&\leq \int_{0}^{1-\sigma} |f(0) - f(x^n)| \, dx + \int_{1-\sigma}^{1} |f(0) - f(x^n)| \, dx \\
&\leq \int_{0}^{1-\sigma} \frac{\varepsilon}{2} \, dx + \int_{1-\sigma}^{1} 2M \, dx < \int_{0}^{1} \frac{\varepsilon}{2} \, dx + 2M\sigma < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\end{align*}
\]
3. Let \( \{b_n\} \) be a sequence of real numbers such that
\[
\sum_{n=1}^{\infty} |b_n| = 1
\]
and let \( f(x) \) be the function given by
\[
\sum_{n=1}^{\infty} b_n \cos(nx).
\]
Prove that the series converges and that \( f \) is continuous on all of \( \mathbb{R} \). Is \( f \) uniformly continuous?

Since \( |\cos(nx)| \leq 1 \) for all \( x \), we have that the series for \( f(x) \) converges uniformly on all of \( \mathbb{R} \) by the Weierstrass \( M \)-test. Moreover, since \( f \) is periodic with period \( 2\pi \), we get uniform continuity for \( f \) from the fact that it is continuous on any closed interval of length greater than \( 2\pi \).

4. Let \( f : X \to Y \) be a continuous mapping from the metric space \( (X, d_X) \) to the metric space \( (Y, d_Y) \). For \( A \subset X \) be a subset of \( X \) we write \( f(A) \) for \( \{y \in Y \mid y = f(a) \text{ for some } a \in A\} \), and for \( B \subset Y \) a subset of \( Y \) we write \( f^{-1}(B) \) for \( \{x \in X \mid f(x) \in B\} \). For each of the following, give a proof or a counterexample:

(a) If \( A \subset X \) is connected, then \( f(A) \subset Y \) is also connected.
(b) If \( B \subset Y \) is connected, then \( f^{-1}(B) \subset X \) is also connected.
(c) If \( A \subset X \) is sequentially compact, then \( f(A) \subset Y \) is also sequentially compact.
(d) If \( B \subset Y \) is sequentially compact, then \( f^{-1}(B) \subset X \) is also sequentially compact.

(a) This is true (and a generalization of the intermediate-value theorem). If \( P \) and \( Q \) were disjoint open subsets of \( Y \) such that \( (P \cap f(A)) \cup (Q \cap f(A)) = f(A) \), then their inverse images would be disjoint open sets in \( X \) such that \( A \cap \left( f^{-1}(P) \cup f^{-1}(Q) \right) = A \) so that \( A \) would not be connected either.

(b) This is false. Consider \( f(x) = x^2 \) on \( \mathbb{R} \), and the set \( B = [1, \infty) \). We have \( f^{-1}(B) = (-\infty, -1] \cup [1, \infty) \) which is not connected.
(c) True. Let \( \{y_n\} \) be a sequence in \( f(A) \), then there is a sequence \( \{x_n\} \) in \( A \) with \( f(x_n) = y_n \) for each \( n \). Since \( A \) is sequentially compact, there is a subsequence of the \( x_n \)'s that converges (say to \( x \)). But then \( f \) of that subsequence will converge in \( f(A) \), since if \( x_n \to x \) in \( A \), then \( f(x_n) \to f(x) \in f(A) \) by continuity. So \( \{y_n\} \) has a convergent subsequence, and so \( f(A) \) is sequentially compact as well.

(d) This is false. Consider \( f(x) \) is the constant 0. Then \( f^{-1}(\{0\}) = \mathbb{R} \) which is not sequentially compact.

5. Consider the set \( \mathbb{Q} \) of rational numbers with its usual metric.

(a) Is every closed, bounded subset of \( \mathbb{Q} \) sequentially compact?

(b) Show that every continuous function \( f: \mathbb{R} \to \mathbb{Q} \) is constant.

Justify your assertions.

(a) No – the set of rational numbers satisfying \( 0 \leq r \leq \sqrt{2} \) is closed in \( \mathbb{Q} \), but there is a sequence of rationals \( \{r_n\} \) in the interval such that \( r_n \to \sqrt{2} \). So this is a Cauchy sequence in \( \mathbb{Q} \cap [0, \sqrt{2}] \) without a limit.

(b) Since \( \mathbb{R} \) is connected, we would have \( f(\mathbb{R}) = \mathbb{Q} \) is connected. But the only connected subsets of \( \mathbb{Q} \) are isolated points, since for any \( r_1 < r_2 \) in \( \mathbb{Q} \), there is an irrational number \( x \) such that \( r_1 < x < r_2 \) and the open sets \( (-\infty, x) \cap \mathbb{Q} \) and \( (x, \infty) \cap \mathbb{Q} \) are a decomposition of \( \mathbb{Q} \) into two disjoint open sets (whose inverse images would be such a decomposition of \( \mathbb{R} \)).

6. Suppose \( f \) is a twice-differentiable function which satisfies the differential equation

\[
\frac{d^2f}{dx^2} = (2 + e^{-x}) f(x)^2
\]

for \( x \geq 0 \). Suppose \( f(0) = 1 \) and \( f'(0) = 0 \). (Do not attempt to solve the equation.) Sketch the graph of \( f \) and show that \( f(x) = 0 \) for one and only one positive value of \( x \).

Certainly \( f''(x) \leq 0 \) for all \( x > 0 \), and \( f''(0) < 0 \). Therefore \( f'(x) < 0 \) for all \( x > 0 \) by the mean-value theorem. (So the graph sketch would look something like the right half of the parabola \( y = 1 - x^2 \).) Moreover, the graph of \( f \) lies below its tangent lines, which have negative slope, so there must be a \( x > 0 \) where \( f(x) = 0 \). There can’t be two of them or else we’d have \( f'(c) \geq 0 \) for some \( c \) between the two solutions.
7. Suppose $f$ is a continuous function defined on the whole real line which is periodic with period one (so $f(x + 1) = f(x)$ for all real $x$). Suppose

$$\int_0^1 f(x) \, dx = 1 \quad \text{and} \quad f(0) = 2.$$ 

Compute the limits

$$\lim_{c \to \infty} \int_0^1 f(cx) \, dx \quad \text{and} \quad \lim_{c \to 0} \int_0^1 f(cx) \, dx$$

and justify your answers.

First, make the change of variables $u = cx$ (or $x = u/c$) so $dx = du/c$ and

$$\int_0^1 f(cx) \, dx = \frac{1}{c} \int_0^c f(u) \, du.$$ 

The limit of this as $c \to 0$ is $f(0)$ by L’Hospital’s rule and the fundamental theorem of calculus (among other reasons).

For the limit as $c \to \infty$, use the fact that $f$ is periodic to conclude first that $f$ is bounded, say $|f(x)| < M$ for all $x \in \mathbb{R}$ and that the integral of $f$ over any interval of length 1 is 1. In particular, if $n \leq c < n + 1$, then (using that same change of variables)

$$\left| \int_0^c f(u) \, du - n \right| = \left| \int_n^c f(u) \, du + \int_0^n f(u) \, du - n \right| = \left| \int_n^c f(u) \, du \right| \leq \int_n^{n+1} M \, du = M.$$ 

Therefore (since $c - 1 < n$ and $c + 1 > n$),

$$\frac{c - 1 - M}{c} < \frac{1}{c} \int_0^c f(u) \, du < \frac{c + 1 + M}{c}.$$ 

The limits of both expressions on the ends is 1 as $c \to \infty$, so the middle one approaches 1 as well, which allows us to conclude that

$$\lim_{c \to \infty} \int_0^1 f(cx) \, dx = 1.$$

8. Say $a_n > 0$ are a sequence of positive real numbers and $a_n \to A$ as $n \to \infty$. Either prove that $A \geq 0$ or provide a counterexample.

If $A$ were negative, there would be an $N$ such that $|a_n - A| < \frac{1}{2}|A|$ for all $n > N$. But then

$$a_n < A + \frac{|A|}{2} = \frac{A}{2} < 0,$$

a contradiction of the fact that $a_n > 0$ for all $n$. Therefore $A$ cannot be negative.
9. For each of the following, give either a proof or a counterexample.

(a) Let \( f \) be a continuous real-valued function on the open interval \( 0 < x < 3 \). Must \( f \) be uniformly continuous on the open interval \( 1 < x < 2 \)?

(b) Suppose instead that \( f \) is only assumed to be continuous on the open interval \( 0 < x < 2 \). Must \( f \) be uniformly continuous on the open interval \( 1 < x < 2 \)?

(a) Yes. Since \( f \) is continuous on \( (0, 3) \) it is continuous on \( [1, 2] \). And a continuous function on a closed, bounded interval is uniformly continuous on that interval. So \( f \) is uniformly continuous on \( [1, 2] \) and thus on \( (1, 2) \).

(b) No. The function \( f(x) = \frac{1}{2 - x} \) is a counterexample (a uniformly continuous function would be bounded).

10. Write the equivalent integral equation formulation for the initial-value problem \( y' = y, \ y(0) = 1 \). Then carry out the first few iterations of the contraction mapping proof of existence and show explicitly that they converge to the solution of the initial-value problem.

The integral formulation is

\[
y(x) = 1 + \int_0^x y(t) \, dt.
\]

So if \( y_0(x) \equiv 1 \), then

\[
y_1(x) = 1 + \int_0^x 1 \, dt = 1 + x
\]

\[
y_2(x) = 1 + \int_0^x (1 + t) \, dt = 1 + x + \frac{x^2}{2}
\]

\[
y_3(x) = 1 + \int_0^x \left(1 + t + \frac{t^2}{2}\right) \, dt = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}
\]

and you can see that (and by induction)

\[
y_n = \sum_{k=0}^{n} \frac{x^k}{k!}
\]

which is the \( n \)th partial sum of the series for \( e^x \). This series converges uniformly to \( e^x \) on any interval \( |x| \leq M \).