MATH 360 –Practice Problems for Exam 2

"Due" Monday, November 6, 2017

1. Is the function $f(x) = \sqrt{x}$ uniformly continuous from $[0, \infty) \to [0, \infty)$?Prove your assertion.

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2. Let $f(x)$ be a continuously differentiable function from $[0, 1]$ to $\mathbb{R}$ and assume that $f(0) = 0$. Show that

$$\int_0^1 |f(x)|^2 \, dx \leqslant \int_0^1 |f'(x)|^2 \, dx$$

as follows:

(a) First, explain why it is true, for each x. that

$$|u(x)|^2 \leqslant \left(\int_0^1 |u'(t)| \, dt \right)^2$$

(b) Now use some famous inequality (ahem) on the right side to get

$$|u(x)|^2 \leqslant \int_0^1 |u'(t)|^2 \, dt$$

(c) Now Integrate both sides of that inequality as x goes from 0 to 1 (note that the right side is a *constant*).

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3. Prove that

$$I_n = \int_0^1 (1 - x^2)^n \, dx = \frac{2^{2n}(n!)^2}{(2n+1)!}$$

(Hint: Induction? Integration by parts?) Can you calculate $\lim_{n \to \infty} I_n$?

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4. Let f be a continuous function on $[a, b]$. Prove that there exists a number x in $[a, b]$ such that

$$\int_a^x f(t) \, dt = \int_x^b f(t) \, dt.$$

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5. Prove that $\cos x \geqslant 1 - \frac{1}{2}x^2$ for all x.

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6. Suppose that

$$6 + f(x) = 2f(-x) + 3x^2 \int_{-1}^1 f(t) \, dt$$

What is $\displaystyle\int_{-1}^1 f(x) \, dx$?

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7. Suppose that $f\colon \mathbb{R} \to \mathbb{R}$ is n times differentiable and that

$$a < x_1 < x_2 < \cdots < x_n < b.$$

Suppose that $p(x)$ is a polynomial of degree $n-1$ with $p(x_j) = f(x_j)$ for $j = 1, \ldots, n$ (we say that p *interpolates* f at the points $x_1, x_2, \ldots, x_n$). The point here is to estimate the error

$$E(t) = f(t) - p(t)$$

for $t \in [a, b]$. Somce $E(x_j) = 0$ for each j, we'll assume that t, x_1,...,x_n are distinct.

Fix t and (in the spirit of finding error estimates for integral approximations) consider the function

$$H(x) = f(x) - p(x) - E(t) \prod_{j=1}^{n} \frac{x - x_j}{t - x_j}$$

(the last term has the product of n fractions).

(a) Show that $H = 0$ at $x = t, x_1, \ldots, x_n$, so H vanishes at $n+1$ points in (a, b).

(b) Show that $H^{(n)}$ (the nth derivative of H) vanishes at some point $c \in (a, b)$.

(c) Now compute $H^{(n)}$ explicitly and deduce that, if $|f^{(n)}(x)| < M$ for all $x \in [a, b]$, then

$$|f(t) - p(t)| \leqslant \frac{M}{n!} \prod_{j=1}^{n} |t - x_j|$$

for all $t \in [a, b]$.