1. Is the function \( f(x) = \sqrt{x} \) uniformly continuous from \([0, \infty) \rightarrow [0, \infty)\)? Prove your assertion.

2. Let \( f(x) \) be a continuously differentiable function from \([0, 1]\) to \( \mathbb{R} \) and assume that \( f(0) = 0 \). Show that
\[
\int_0^1 |f(x)|^2 \, dx \leq \int_0^1 |f'(x)|^2 \, dx
\]
as follows:

(a) First, explain why it is true, for each \( x \), that
\[
|u(x)|^2 \leq \left( \int_0^1 |u'(t)| \, dt \right)^2
\]

(b) Now use some famous inequality (ahem) on the right side to get
\[
|u(x)|^2 \leq \int_0^1 |u'(t)|^2 \, dt
\]

(c) Now Integrate both sides of that inequality as \( x \) goes from 0 to 1 (note that the right side is a constant).

3. Prove that
\[
I_n = \int_0^1 (1 - x^2)^n \, dx = \frac{2^{2n}(n!)^2}{(2n + 1)!}
\]
(Hint: Induction? Integration by parts?) Can you calculate \( \lim_{n \to \infty} I_n \)?

4. Let \( f \) be a continuous function on \([a, b]\). Prove that there exists a number \( x \) in \([a, b]\) such that
\[
\int_a^x f(t) \, dt = \int_x^b f(t) \, dt.
\]

5. Prove that \( \cos x \geq 1 - \frac{1}{2}x^2 \) for all \( x \).

6. Suppose that
\[
6 + f(x) = 2f(-x) + 3x^2 \int_{-1}^1 f(t) \, dt
\]
What is \( \int_{-1}^1 f(x) \, dx \)?
7. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is $n$ times differentiable and that

$$a < x_1 < x_2 < \cdots < x_n < b.$$ 

Suppose that $p(x)$ is a polynomial of degree $n - 1$ with $p(x_j) = f(x_j)$ for $j = 1, \ldots, n$ (we say that $p$ interpolates $f$ at the points $x_1, x_2, \ldots, x_n$). The point here is to estimate the error

$$E(t) = f(t) - p(t)$$

for $t \in [a, b]$. Since $E(x_j) = 0$ for each $j$, we’ll assume that $t, x_1, \ldots, x_n$ are distinct.

Fix $t$ and (in the spirit of finding error estimates for integral approximations) consider the function

$$H(x) = f(x) - p(x) - E(t) \prod_{j=1}^{n} \frac{x - x_j}{t - x_j}$$

(the last term has the product of $n$ fractions).

(a) Show that $H = 0$ at $x = t, x_1, \ldots, x_n$, so $H$ vanishes at $n + 1$ points in $(a, b)$.

(b) Show that $H^{(n)}$ (the $n$th derivative of $H$) vanishes at some point $c \in (a, b)$.

(c) Now compute $H^{(n)}$ explicitly and deduce that, if $|f^{(n)}(x)| < M$ for all $x \in [a, b]$, then

$$|f(t) - p(t)| \leq \frac{M}{n!} \prod_{j=1}^{n} |t - x_j|$$

for all $t \in [a, b]$. 