Math 360: Series of constants and functions

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A **series** is an infinite sum, i.e., the sum of a sequence. To make sense of this, given the sequence

\[
\{a_n\} = a_1, a_2, a_3, \ldots
\]

we form a new sequence – the **sequence of partial sums** \(\{s_n\}\) where

\[
s_1 = a_1 \\
s_2 = a_1 + a_2 \\
s_3 = a_1 + a_2 + a_3 \\
\vdots \\
s_n = a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{n} a_k
\]
If the sequence of partial sums $s_n$ has a limit, say $\lim_{n \to \infty} s_n = S$, then we say that “the series converges to $S$” or “the sum of the series is $S$, and write

$$\sum_{k=1}^{\infty} a_k = \lim_{n \to \infty} \sum_{k=1}^{n} a_k = S.$$ 

Otherwise, the series **diverges**.

For instance, $\sum_{k=1}^{\infty} \frac{1}{2^k} = 1$ and $\sum_{k=1}^{\infty} \frac{1}{n!} = e - 1$. 
Cauchy criterion

We can bring to bear everything we already know about sequences to series via the sequence of partial sums.

For instance, a series is **Cauchy** if the sequence of partial sums is, in other words given any $\varepsilon > 0$ there is an $N > 0$ such that for all $m, n > N$ we have

$$|s_n - s_m| = \left| \sum_{k=m+1}^{n} a_k \right| = |a_{m+1} + a_{m+2} + \cdots + a_n| < \varepsilon$$

A series is convergent if and only if it is Cauchy.

In particular (with $n = m + 1$): If a series is convergent, then $\lim_{n \to \infty} a_n = 0$. (nth term test). The harmonic series shows that the converse is not true.
Series with positive terms

If \( a_k > 0 \) for each \( k \), then the sequence of partial sums \( s_n \) is monotonically increasing. By the fundamental axiom, the sequence \( \{s_n\} \) and hence the series \( \sum_{k=0}^{\infty} a_k \), will converge if there is an upper bound to the sequence of partial sums. This observation is at the root of many of the familiar “tests” for convergence:

**Comparison test**

Suppose \( b_k \geq a_k \geq 0 \) for every \( k \).

(a) If \( \sum b_k \) converges then so does \( \sum a_k \)

(b) If \( \sum a_k \) diverges then so does \( \sum b_k \)

In both cases the partial sums for one series bound those of the other, either from above or below.
Suppose $f(x) \geq 0$ is a decreasing function from $[0, \infty)$ to $\mathbb{R}$. Then the series $\sum_{n=1}^{\infty} f(n)$ converges if and only if

$$\lim_{M \to \infty} \int_{1}^{M} f(x) \, dx$$

exists and is finite.

We usually write that limit as the “improper integral” $\int_{1}^{\infty} f(x) \, dx$.

Proof: Compare $\sum_{n=1}^{\infty} f(n)$ to $\sum_{n=1}^{\infty} \int_{n-1}^{n} f(x) \, dx$ (for convergence) or to $\sum_{n=1}^{\infty} \int_{n}^{n+1} f(x) \, dx$ (for divergence).
Ratio test

For the series $\sum_{k=1}^{\infty} a_k$ (assume $a_k > 0$ for the time being), suppose

$$\lim_{k \to \infty} \frac{a_{k+1}}{a_k} = L.$$ 

(a) If $L < 1$ then the series converges.
(b) If $L > 1$ then the series diverges.
(c) If $L = 1$ then no conclusion can be drawn.

Proof: Comparison with the geometric series $\sum r^n$ for $r$ between $L$ and 1.

Some other tests (limit comparison test, root test) will be discussed in the homework.
A series of the form $\pm \sum_{k=1}^{\infty} (-1)^{k+1}a_k$ (with $a_k > 0$) is called alternating.

**Alternating series test**

If $a_k$ is a positive monotonically decreasing sequence then

$$\sum_{k=1}^{\infty} (-1)^{k+1}a_k$$

converges if and only if $a_k \to 0$.

Proof: Consider the subsequence $s_{2n}$ – it’s monotonically increasing and bounded above by $a_1$. Since $a_n \to 0$, the subsequence $s_{2n+1}$ converges to the same limit. But then so does all of $s_n$. 
Absolute convergence

Say that a series $\sum a_n$ of arbitrary numbers converges \textbf{absolutely} if $\sum |a_n|$ converges.

Using the Cauchy criterion, an absolutely convergent series is itself convergent.

So we can extend the ratio test by considering $\lim |a_{n+1}/a_n|$. 
Now we’ll begin consideration of sequences and series of functions (pay attention to whether it’s a sequence or a series!).

**Definition: (pointwise convergence)**

A sequence \( \{f_n\} \) or series \( \sum f_n \) of functions **converges pointwise** to \( F \) in the interval from \( a \) to \( b \) (open or closed or otherwise) if for every \( x \) in the interval the numerical sequence \( \{f_n(x)\} \) or series \( \sum f_n(x) \) converges to \( F(x) \).

Think about \( \{e^{-nx}\} \) on \([0, \infty)\) or \( \{\arctan nx\} \) on all of \( \mathbb{R} \). (Pointwise limits of continuous functions need not be continuous).

Or the sequence \( \{(n + 1)x^n\} \) on \([0, 1)\). (The integral of the limit is not the limit of the integrals).
We need a stronger notion of convergence to preserve desirable properties:

**Definition: (uniform convergence)**

A sequence $f_n$ of functions on a domain $D$ **converges uniformly** to $F$ if for any $\varepsilon > 0$ there is an $N > 0$ (depending only on $\varepsilon$ and not on $x \in D$) such that

$$|F(x) - f_n(x)| < \varepsilon \quad \text{for all } n \geq N \text{ and all } x \in D$$

Define the notion of “uniformly Cauchy” and show that a uniformly Cauchy sequence is uniformly convergent.

As we shall see, the limit of a uniformly convergent sequence of continuous functions is continuous. Likewise for integrable, and the integral of the limit is equal to the limit of the integrals.