Math 360: The real numbers

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Some cautionary tales: $\mathbb{R}$ vs $\mathbb{Q}$

"Analysis consists of the difficult proofs of obvious theorems"

Are the answers intuitive:

- What is the (general) solution of $g'(t) = t^2$? Are you sure?
- Are the "constant value theorem" and the "intermediate value theorem" obvious?
- If $f'(t) > 0$ for all t does $a > b$ imply that $f(a) > f(b)$?

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Over $\mathbb{Q}$, all of these are neither intuitive nor true! So what makes the real numbers "better" for analysis?

Fields

Algebraic axioms first: A field $\mathbb{F}$, or $(\mathbb{F}, +, \times)$ is a set $\mathbb{F}$ with two binary operations such that:

- (A1) $a + (b + c) = (a + b) + c$ and $a + b = b + a$.
- (A2) There is a unique element $0 \in \mathbb{F}$ with $a + 0 = a$ for all $a \in \mathbb{F}$.
- (A3) For each $a \in \mathbb{F}$ there is $-a \in \mathbb{F}$ with $a + (-a) = 0$.

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So $(\mathbb{F}, +)$ is a commutative group; write $a - b$ for $a + (-b)$. Also write ab for $a \times b$. Then

- (M1) $a(bc) = (ab)c$ and $ab = ba$
- (M2) There is a unique $1 \in \mathbb{F}$ with $a1 = a$ for all $a \in \mathbb{F}$.
- (M3) For each $a \in \mathbb{F}$ with $a \neq 0$ there is a^{-1} with $aa^{-1} = 1$.

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- (D1) $a(b + c) = ab + ac$ (distributive law)

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So far, $\mathbb{F}$ is a field. Now for the order axioms.

There is a nonempty subset $\mathbb{P} \subset \mathbb{F}$ and write $a > 0$ iff $a \in \mathbb{P}$:

- (P1) If $a \in \mathbb{P}$ and $b \in \mathbb{P}$ then $a + b \in \mathbb{P}$ and $ab \in \mathbb{P}$.
- (P2) For all $a \in \mathbb{F}$ either $a \in \mathbb{P}$ or $-a \in \mathbb{P}$ or $a = 0$. (Trichotomy)

Write $a > b$ iff $a - b \in \mathbb{P}$ and write $a \geq b$ iff $a > b$ or $a = b$.

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Write $a > b$ iff $a - b \in \mathbb{P}$ and write $a \geq b$ iff $a > b$ or $a = b$.

Now $\mathbb{F}$ is an ordered field. $\mathbb{R}$ and $\mathbb{Q}$ are examples of ordered fields (there are others).

Exercise: Show that $a^2 = (-a)^2$, so that $a^2 > 0$ for all $a \neq 0$, and so $1 > 0$. And by induction $n > 0$ for every positive integer $n = 1 + 1 + \cdots + 1$.

Absolute value

Definition of absolute value

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{otherwise} \end{cases}$$

Lemma 1

1. $|a| \geq 0$.
2. $|(|a|)| = |a|$.
3. $|-a| = |a|$
4. $|ab| = |a||b|$
5. $|a| + |b| \geq |a + b|$

Exercise: All of these except the last one are obvious. Prove the last one (well, maybe also the next-to-last one).

Limit of a sequence

We work in the ordered field $\mathbb{F}$ (think $\mathbb{R}$ or $\mathbb{Q}$, but there are others).

Definition

The sequence $a_1, a_2, \ldots$ tends to the limit L as n tends to infinity (or $a_n \to L$ as $n \to \infty$, or $\lim_{n\to\infty} a_n = L$) if, given any $\varepsilon > 0$, we can find an integer $N(\varepsilon)$ such that

$$|a_n - L| < \varepsilon \quad \text{for all} \quad n > N(\varepsilon).$$

Properties of limits of sequences

Lemma 2

Suppose $a_n \to L$ as $n \to \infty$. Then:

1. The limit is unique (i.e., if also $a_n \to M$ then $L = M$).
2. (Subsequences) If $n(1) < n(2) < n(3) < \cdots$ then $a_{n(j)} \to L$ as $j \to \infty$.
3. If $a_n = c$ for all n, then $L = c$.
4. If $b_n \to M$ as $n \to \infty$ then $a_n + b_n \to L + M$ and $a_n b_n \to LM$.
5. If $a_n \neq 0$ for each n and if $L \neq 0$, then $a_n^{-1} \to L^{-1}$
6. If $a_n \leq A$ for each n, then $L \leq A$. Likewise if $a_n \geq A$ for each n then $L \geq A$.

Epsilontics; proof of lemma

We'll give careful proofs of the properties on the previous slide to illustrate "epsilontics".

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1. Suppose $a_n \to L$ as $n \to \infty$. Then the limit is unique (i.e., if also $a_n \to M$ then $L = M$).

From the definition, we know that given $\varepsilon > 0$ we can find $N_1(\varepsilon)$ so that $|a_n - L| < \varepsilon$ for all $n > N_1(\varepsilon)$, and we can find $N_2(\varepsilon)$ so that $|a_n - M| < \varepsilon$ for all $n > N_2(\varepsilon)$. If it were the case that $L \neq M$, then picking $\varepsilon = \frac{1}{2}|L - M|$ we have $\varepsilon > 0$, and then choosing any $n > \max(N_1(\varepsilon), N_2(\varepsilon))$ we have:

$$|L - M| = |(L - a_n) + (a_n - M)| \leq |a_n - L| - |a_n - M| < \varepsilon + \varepsilon = |L - M|.$$

This is impossible, so (by contradiction, or *reductio ad absurdam*) we must have $L = M$.

Proof of lemma

2. (Subsequences) Suppose $a_n \to L$ as $n \to \infty$. If $n(1) < n(2) < n(3) < \cdots$ then $a_{n(j)} \to L$ as $j \to \infty$.

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Given $\varepsilon > 0$ we can find an $N(\varepsilon)$ such that $|a_n - L| < \varepsilon$ for all $n > N(\varepsilon)$. Since $n(j) \geq j$ (proof by induction), we also have $|a_{n(j)} - L| < \varepsilon$ for all $j > N(\varepsilon)$.

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3. If $a_n = c$ for all n, then $a_n \to c$ as $n \to \infty$.

Given $\varepsilon > 0$ we can take $N(\varepsilon)$ to be equal to 1, since

$$|a_n - c| = 0 < \varepsilon$$

for all n.

Limits of sums

4. If $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = M$ then $\lim_{n\to\infty} a_n + b_n = L + M$.

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By hypothesis, given $\varepsilon > 0$ we can find $N_1(\varepsilon)$ such that $|a_n - L| < \varepsilon$ for all $n > N_1(\varepsilon)$ and we can find $N_2(\varepsilon)$ such that $|b_n - M| < \varepsilon$ for all $n > N_2(\varepsilon)$. Now let $N_3(\varepsilon) = \max(N_1(\varepsilon/2), N_2(\varepsilon/2))$. Then for $n > N_3(\varepsilon)$ we have

$$\begin{aligned} |(a_n + b_n) - (L + M)| &= |(a_n - L) + (b_n - M)| \\ &\leq |a_n - L| + |b_n - M| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

which gives the result.

Limits of products

4. If $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = M$ then $\lim_{n\to\infty} a_n b_n = LM$.

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By hypothesis, given $\varepsilon > 0$ we can find $N_1(\varepsilon)$ such that $|a_n - L| < \varepsilon$ for all $n > N_1(\varepsilon)$ and we can find $N_2(\varepsilon)$ such that $|b_n - M| < \varepsilon$ for all $n > N_2(\varepsilon)$. Now let $N_3(\varepsilon) = \max(N_1(1), N_1(\varepsilon/(2(|M|+1))), N_2(\varepsilon/(2(|L|+1))))$ (because it is possible that $M = 0$). Then for $n > N_3(\varepsilon)$ we have (noting that $|a_n| < |L| + 1$)

$$\begin{aligned} |a_n b_n - LM| &= |a_n b_n - a_n M + a_n M - LM| \\ &\le |a_n||b_n - M| + |M||a_n - L| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

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Limits of reciprocals

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Let $\varepsilon > 0$ be given. By hypothesis, given $\varepsilon > 0$ we can find $N_1(\varepsilon)$ such that $|a_n - L| < \varepsilon$ for all $n > N_1(\varepsilon)$. Since $L \neq 0$ we also have $\frac{1}{2}|L| \neq 0$, so set $N(\varepsilon) = \max(N_1(\frac{1}{2}L), N_1(\frac{1}{2}L^2\varepsilon))$. If $n > N(\varepsilon)$, then

$$\begin{aligned}\left|\frac{1}{a_n} - \frac{1}{L}\right| &= \frac{|L - a_n|}{|L||a_n|} \\ &\leq \frac{2|L - a_n|}{|L|^2} < \varepsilon\end{aligned}$$

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Limits with bounds

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Suppose $L > A$. Then set $N = n(L - A)$. If $n > N$, then

$$a_n = (a_n - L) + L \geq L - |a_n - L| > L - (L - A) = A$$

which is a contradiction.

Functions and continuity

We're still working in an ordered field $\mathbb{F}$. Let $E \subset \mathbb{F}$ and $f\colon E \to F$ be a function from E to $\mathbb{F}$.

Definition

f is continuous at $x \in E$ if given any $\varepsilon > 0$ we can find $\delta(\varepsilon, x) > 0$ such that

$$|f(x) - f(y)| < \varepsilon \quad \text{for all } y \in E \text{ such that} \quad |x - y| < \delta(\varepsilon, x).$$

If f is continuous at every point of E then say that $f\colon E \to \mathbb{F}$ is a continuous function.

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Lemma 3 (continuity and sequences)

If f is continuous at $x \in E$ and a_n is a sequence of points of E with $a_n \to x$ as $n \to \infty$, then $f(a_n) \to f(x)$ as $n \to \infty$.

Continuity properties

Lemma 4

Let E be a subset of the ordered fiend $\mathbb{F}$ and f and g are functions from E to $\mathbb{F}$.

1. If $f(x) = c$ and $g(x) = x$ for all $x \in E$ then f and g are continuous on E.
2. Define $f + g \colon E \to \mathbb{F}$ via $(f + g)(x) = f(x) + g(x)$ and $f \times g \colon E \to \mathbb{F}$ via $(f \times g)(x) = f(x)g(x)$ for all $x \in E$. If f and g are continuous at $y \in E$ then so are $f + g$ and $f \times g$.
3. If $f(x) \neq 0$ for all $x \in E$. If f is continuous at $y \in E$, then so is $1/f$ (defined in the obvious way).

Corollary 5

Polynomials are continuous on $\mathbb{F}$, and rational functions are continuous on subsets $E \subset \mathbb{F}$ on which the denominator never vanishes.

A cautionary example

First, note that the equation $x^2 = 2$ has no solutions in $\mathbb{Q}$.

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Let $f : \mathbb{Q} \to \mathbb{Q}$ be defined via:

$$f(x) = \begin{cases} -1 & \text{if} \quad x^2 < 2 \\ 1 & \text{if} \quad x^2 > 2 \end{cases}$$

Then f is continuous on $\mathbb{Q}$ but none of the "intuitive" theorems about continuous functions ("constant value theorem", intermediate value theorem) hold for f, and the "positive derivative implies increasing" theorem does not hold for $g(x) = x + f(x)$.

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What property to we have to add to distinguish the real numbers from the rationals?

The fundamental axiom

The fundamental axiom

If $a_n \in \mathbb{R}$ and $a_1 \leq a_2 \leq a_3 \leq \cdots$ and if there is some number $A \in \mathbb{R}$ such that $a_n < A$ for all n, then there is an $L \in \mathbb{R}$ such that $a_n \to L$ as $n \to \infty$.

In other words, every increasing sequence that is bounded above tends to a limit.

Pretty much all of classical real analysis can be obtained from the algebraic properties of $\mathbb{R}$ together with this axiom. There are several other equivalent statements, equivalent to the axiom, which we will prove. But we could have used any of them as our fundamental axiom (and the textbook does this).

The Archimedean axiom

It's called an axiom but we're going to prove it, obvious as it seems:

Theorem 6 (Axiom of Archimedes)

$$\frac{1}{n} \to 0 \quad \text{as} \quad n \to \infty$$

Proof: $1/n$ is a decreasing sequence bounded below by 0 so it has a limit, L. Then multiplying the sequence $1/n$ by the constant sequence all of whose terms are $1/2$ gives that the limit of the sequence $(1/2) \times (1/n) = 1/(2n)$ is $L/2$. But $1/(2n)$ is also a subsequence of $1/n$, so its limit must also be L. Therefore $L = L/2$ which implies $L = 0$.

Corollary 7

There is no $M \in \mathbb{R}$ such that $M > n$ for all $n \in \mathbb{Z}$.

$\mathbb{Q}$ is dense in $\mathbb{R}$

(a) Using the fact that every non-empty set of integers bounded above has a maximum (with proof!), show that if $x \in \mathbb{R}$, there is an integer m such that $m \leq x < m + 1$ and so in particular $|x - m| < 1$.

(b) Use this to show that if $x \in \mathbb{R}$ and q is a positive integer, then there is an integer p such that $\left|x - \frac{p}{q}\right| < \frac{1}{n}$.

(c) If $x \in \mathbb{R}$, then given any $\varepsilon > 0$ there exists $y \in \mathbb{Q}$ such that $|x - y| < \varepsilon$. (In other words, $\mathbb{Q}$ is dense in $\mathbb{R}$.

The intermediate value theorem

Theorem 8 (Intermediate value theorem)

We work in $\mathbb{R}$. If $f\colon [a,b] \to \mathbb{R}$ is continuous, and $f(a) \le 0 \le f(b)$ then there exists $c \in [a,b]$ such that $f(c) = 0$.

Proof: (by the "bisection method" or "lion hunting"). Define two sequences a_n and b_n inductively as follows: Let $a_0 = a$ and $b_0 = b$. Then set $c_0 = \frac{1}{2}(a_0 + b_0)$. Either $f(c_0) \ge 0$ in which case set $a_1 = a_0$ and $b_1 = c_0$, or else $f(c_0) < 0$ in which case set $a_1 = c_0$ and $b_1 = b_0$. Either way, we have

$$f(a_1) \le 0 \le f(b_1) \qquad a_0 \le a_1 \le b_1 \le b_0 \qquad b_1 - a_1 = \frac{1}{2}(b_0 - a_0)$$

Continue inductively, so that

$$f(a_n) \le 0 \le f(b_n) \quad a_{n-1} \le a_n \le b_n \le b_{n-1} \quad b_n - a_n = \frac{1}{2}(b_{n-1} - a_{n-1})$$

for all $n \ge 1$.

Proof of the intermediate value theorem (continued)

Theorem 8 (Intermediate value theorem)

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By their definition, we have $a_0 \leq a_1 \leq a_2 \cdots$ so the a_n are an increasing sequence that is bounded above by $b_0 = b$. Therefore there is a number c so that $a_n \to c$ as $n \to \infty$. We also have $a \leq c \leq b$. We can also note that $b_n - a_n = 2^{-n}(b_0 - a_0)$, so that $b_n - a_n \to 0$ (by Archimedes) and so

$$b_n = a_b + (b_n - a_n) \to c + 0 = c$$

as $n \to \infty$.

Since f is continuous, we have $f(a_n) \to f(c)$ as $n \to \infty$. Since $f(a_n) \leq 0$ for all n, we have $f(c) \leq 0$. Likewise, $f(b_n) \to f(c)$ and $f(b_n) \geq 0$, so we also have $f(c) \geq 0$. Therefore $f(c) = 0$, completing the proof.

The derivative

Definition of the derivative

Let $f : (a, b) \to \mathbb{R}$ (where (a, b) is the open interval $a < x < b$). f is differentiable at $x \in (a, b)$ with derivative $f'(x)$ if, given $\varepsilon > 0$, we can find $\delta(x, \varepsilon) > 0$ such that $a < x - \delta(x, \varepsilon) < x + \delta(x, \varepsilon) < b$ and

$$\left| \frac{f(y) - f(x)}{y - x} - f'(x) \right| < \varepsilon$$

whenever $0 < |y - x| < \delta(x, \varepsilon)$.

Exercise: If f and g are differentiable at x with derivatives $f'(x)$ and $g'(x)$, and if α and β are real constants, show that $\alpha f + \beta g$ is differentiable at x with derivative $\alpha f'(x) + \beta g'(x)$.

The mean value inequality

Theorem 9 (The mean value inequality)

Suppose $A < a < b < B$, and f is differentiable on (at every point of) (A, B) and that there is a real number M such that $f'(x) \leq M$ for all $x \in (A, B)$. Then

$$f(b) - f(a) \leq (b - a)M.$$

Proof: We assume the contrary, namely that $f(b) - f(a) > M(b - a)$ and derive a contradiction, using bisection. If that were true, then we can find an $\varepsilon > 0$ small enough so that in fact $f(b) - f(a) > (M + \varepsilon)(b - a)$ (we had to get an ε in there somewhere!). We'll show that this is impossible.

So set $a_0 = a$, $b_0 = b$ and let $c_0 = \frac{1}{2}(a_0 + b_0)$. Observe:

$$\Big(f(c_0) - f(a_0) - (M + \varepsilon)(c_0 - a_0)\Big) + \Big(f(b_0) - f(c_0) - (M + \varepsilon)(b_0 - c_0)\Big)$$
$$= \Big(f(b_0) - f(a_0) - (M + \varepsilon)(b_0 - a_0)\Big) > 0$$

Proof of the mean value inequality (continued)

Therefore, one of the two parenthesized expressions on the left must be positive.
If $\Big(f(b_0) - f(c_0) - (M+\varepsilon)(b_0 - c_0)\Big) > 0$ then set $a_1 = c_0$ and $b_1 = b_0$; otherwise set $a_1 = a_0$ and $b_1 = c_0$. Either way we now have:

$$f(b_1) - f(a_1) > (M+\varepsilon)(b_1 - a_1), \quad a_0 \le a_1 \le b_1 \le b_0, \quad b_1 - a_1 = \frac{1}{2}(b_0 - a_0)$$

You can see where this is going: Continue inductively so that

$$f(b_n) - f(a_n) > (M+\varepsilon)(b_n - a_n), \quad a_{n-1} \le a_n \le b_n \le b_{n-1},$$

and $b_n - a_n = \frac{1}{2}(b_{n-1} - a_{n-1})$. So as before, there is a c so that both $a_n \to c$ and $b_n \to c$ as $n \to \infty$.

Proof of the mean value inequality (concluded)

Because f is differentiable at c, and because $a_n \to c$ and $b_n \to c$ we know we can find a $\delta(c, \varepsilon)$ so that

$$\left| \frac{f(y) - f(c)}{y - c} - f'(c) \right| < \varepsilon$$

whenever $0 < |y - c| < \delta(c, \varepsilon)$, and an N so that $b_N - c < \delta(c, \varepsilon)$ and $c - a_N < \delta(c, \varepsilon)$. Therefore

$$\frac{f(b_N) - f(c)}{b_N - c} - f'(c) < \varepsilon \quad \text{and} \quad \frac{f(c) - f(a_N)}{c - a_n} - f'(c) < \varepsilon.$$

Combine these to conclude that

$$f(b_N) - f(a_N) < \left(f'(c) + \varepsilon\right)(b_N - a_N) \leq (K + \varepsilon)(b_N - a_N)$$

which contradicts our assumption about $f(b_N) - f(a_N)$, and completes the proof.

The constant value theorem

Theorem 10 (constant value theorem)

If U is an open interval in $\mathbb{R}$ (or if $U = \mathbb{R}$) and $f\colon U \to \mathbb{R}$ is differentiable with $f'(x) = 0$ for all $x \in U$, then f is a constant function.

Proof:Suppose a and b are in U, with $b > a$. Applying the mean value inequality with $M = 0$ get that $f(b) - f(a) \leq 0$. Then apply it to $-f$ with $M = 0$ to get $f(a) - f(b) \leq 0$, so $f(a) = f(b)$. Since a and b were arbitrary, f is constant.

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Corollary 11

If f and g are differentiable on U and $f'(x) = g'(x)$ for all $x \in U$, then $f(x) = g(x) + c$.

(Apply the preceding to $f - g$.)