Math 360: More on derivatives

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Critical points

Proposition

Let f be differentiable on the interval $[a, b]$, and suppose that the maximum of f on $[a, b]$ occurs at $x = c \in (a, b)$. Then $f'(c) = 0$. (Likewise for the minimum).

Since $f(c) \geq f(x)$ for all $x \in [a, b]$, and the interval $(c - \delta\,,\ c + \delta)$ is contained in (a, b) for some δ, we know that for $x > c$ and $x - c < \delta$ we have

$$\frac{f(x) - f(c)}{x - c} \leq 0$$

and so $f'(c)$, which is the limit of this as $x \to c$, cannot be positive. Likewise, for $x < c$ and $c - x < \delta$ we have

$$\frac{f(x) - f(c)}{x - c} \geq 0$$

and so $f'(c)$ cannot be negative. Therefore $f'(c) = 0$ as claimed.

Critical points

As usual, we'll say that $x = c$ is a *critical point* of f if either $f'(c) = 0$ or $f'(c)$ does not exist. The proposition on the previous slide says that the maximum (and minimum) of f can occur only at a critical point of f or at an endpoint of the interval.

We could apply this observation to do max/min problems, but instead we'll explore a theoretical consequence:

Lemma (Rolle's Theorem)

Let f be differentiable on the interval $[a, b]$ and suppose that $f(a) = f(b)$. Then there is a point $c \in (a, b)$ where $f'(c) = 0$.

Observe that either the maximum or the minimum of c must occur at point of (a, b) (otherwise the max and min are equal, so f is constant and $f'(x) = 0$ for all $x \in (a, b)$). Then apply the proposition on the last slide.

The Mean Value Theorem

Theorem (Mean value theorem)

Let f be differentiable on (a, b) and continuous on $[a, b]$. Then there is a point $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

To prove this, apply Rolle's theorem to the function

$$g(x) = f(x) - \frac{f(b) - f(a)}{b - a}(x - a)$$

Notice that $g(a) = f(a)$ and $g(b) = f(a)$ as well, so Rolle's theorem applies. And if $g'(c) = 0$, we have

$$f'(c) - \frac{f(b) - f(a)}{b - a} = 0$$

which is what the theorem says.

First applications of the MVT

Proposition

If $f'(x) > 0$ on (a, b) then $f(b) > f(a)$. Likewise if $f'(x) < 0$ on (a, b) then $f(b) < f(a)$.
Also, if $f'(x) \geq 0$ on (a, b) then $f(b) \geq f(a)$. Likewise if $f'(x) \leq 0$ on (a, b) then $f(b) \leq f(a)$.

The proofs are all the same and by contradiction. For the first one, suppose $f(b) \leq f(a)$, Then there is a $c \in (a, b)$ where $f'(c) \leq 0$, contradicting the mean value theorem.

Proposition

If $f'(x) = 0$ for all $x \in (a, b)$ then $f(x)$ is constant on $[a, b]$.

If $f(x) \neq f(y)$ for $x, y \in [a, b]$ then there would be a c where $f'(c) \neq 0$ according to the MVT.

The Cauchy Mean Value Theorem

Corollary (that didn't fit on the last slide)

If $f'(x) = g'(x)$ for all $x \in (a, b)$, then $f(x) = g(x) + C$ for a constant C.

Apply the preceding to $f - g$.

Theorem (Cauchy mean value theorem)

Suppose f and g are continuous on $[a, b]$ and differentiable on (a, b), and that $g'(x) \neq 0$ for all $x \in (a, b)$. Then there is a $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

Proof: Apply Rolle's theorem to the function

$$h(x) = f(x) - \frac{f(b) - f(a)}{g(b) - g(a)}\, g(x).$$

L'Hospital's rule

Theorem (L'Hospital's rule)

Suppose that f and g are differentiable and that $g'(x) \neq 0$ on an open interval I that contains $x = a$ (except possibly at a). Further, suppose that

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0.$$

Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

if the limit on the right side exists (or is $\pm\infty$).

If we don't invoke the "except at $x = a$" clause, so $f(a) = g(a) = 0$ and if $f'(a)$ and $g'(a)$ exist and $g'(a) \neq 0$, then the proof is not hard, and is on the next slide.

Proof of L'Hospital's rule, easy case

With all the assumptions at the bottom of the last slide, we have:

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f(x)-f(a)}{g(x)-g(a)}$$

$$= \lim_{x\to a}\frac{\dfrac{f(x)-f(a)}{x-a}}{\dfrac{g(x)-g(a)}{x-a}} = \frac{\displaystyle\lim_{x\to a}\frac{f(x)-f(a)}{x-a}}{\displaystyle\lim_{x\to a}\frac{g(x)-g(a)}{x-a}}$$

$$= \frac{f'(a)}{g'(a)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

More general proof

Assume only that the limits of f and g are zero at $x = a$ and define functions F and G to take the value zero at $x = a$. Then F and G are continuous on the whole interval I. Now let $x > a$ with x still in the interval I – then F and G are continuous on $[a, x]$ and differentiable on (a, x) and $G' \neq 0$ on (a, x), so by the Cauchy MVT there is a y with $a < y < x$ and

$$\frac{F'(y)}{G'(y)} = \frac{F(x) - F(a)}{G(x) - G(a)} = \frac{F(x)}{G(x)}.$$

Now let $x \to a^+$ – since y is trapped between a and x as $x \to a^+$, we get the result for the right-hand limit. Similarly, the left-hand limit gives the predicted result.

If $a = \infty$, get the result by substituting $y = 1/x$ so that $y \to 0^+$. This completes the $0/0$ case of L'Hospital's rule.

Toward Taylor's theorem

Lemma

Let g and h be differentiable on $(-a, a)$, where $a > 0$. If $g'(t) \leq h'(t)$ for all $0 \leq t < a$ and $g(0) = h(0)$ then $g(t) \leq h(t)$ for all $0 \leq t < a$. We also have $h(t) \leq g(t)$ for $-a < t < 0$.

Use the mean value theorem on $u(t) = h(t) - g(t)$: For $t > 0$ we have $u(0) = 0$ and $u'(t) = h'(t) - g'(t) \geq 0$, so $u(t) = h(t) - g(t) \geq 0$ for $0 \leq t < a$ by MVT.

Lemma

If $|f'(t)| \leq |t|^r$ for all $t \in (-a, a)$ and $f(0) = 0$, then $|f(t)| \leq |t|^{r+1}/(r+1)$ for all $|t| < a$.

This follows from the above – we have $f'(t) \leq t^r$ for $0 < t < a$ so use the lemma above with $g(t) = f(t)$ and $h(t) = t^{r+1}/(r+1)$.

Getting to Taylor's theorem

Proposition

If g is n times differentiable on $(-a, a)$ with $|g^{(n)}(t)| < M$ for all $t \in (-a, a)$ and if $g(0) = g'(0) = \cdots = g^{(n-1)}(0) = 0$, then

$$|g(t)| \leq \frac{M|t|^n}{n!}.$$

Theorem (Taylor formula with Lagrange-type remainder)

If f is n times differentiable on $(-a, a)$ with $|f^{(n)}(t)| \leq M$ for all $t \in (-a, a)$ then

$$\left| f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} t^k \right| \leq \frac{M|t|^n}{n!}$$

for all $|t| < a$.