Math 410  Dr. DeTurck  Practice Problems for Midterm 1  October 6, 2009

1. For which values of $z$ is $z^2 = |z|^2$? For which values of $z$ is $z^2 = i|z|^2$?

2. Let $f(z) = z + 1/z$. What is the image of the unit circle under the mapping defined by $f$?

3. On the domain $\{z = x + iy, 0 \leq x \leq 2\pi, 0 \leq y \leq 2\pi\}$, what is the maximum value of $|\cos z|$?

4. Let $u(x, y) = 2x - xy$. Find a function $v(x, y)$ so that
   \[ f(x + iy) = u(x, y) + iv(x, y) \]
   is a holomorphic function. Express $f(z)$ in terms of $z$ alone.

5. Find all the solutions of $\sin z = \sqrt{3}$.

6. Calculate $\int_\gamma z\,dz$, $\int_\gamma \frac{dz}{z}$, where $\gamma$ is the unit circle, traversed once in the counterclockwise direction.

7. Give an example of a (nontrivial) simple closed curve $\gamma$ for which
   \[ \int_\gamma \frac{dz}{z^2 + z + 1} = 0 \]
   and another for which
   \[ \int_\gamma \frac{dz}{z^2 + z + 1} \neq 0. \]
   What is the value of the second integral over your curve?

8. Calculate
   \[ \int_{-\infty}^{\infty} \frac{\cos x}{x^2 + 1}\,dx \]
   by applying the Cauchy Integral Formula to
   \[ \int_\gamma \frac{e^{iz}}{(z + i)(z - i)}\,dz \]
   where $\gamma$ is the “standard” semicircular contour of radius $R$ and letting $R$ go to infinity. Be sure to estimate what happens on the circle part carefully.