1. For which values of \( z \) is \( z^2 = |z|^2 \)? For which values of \( z \) is \( z^2 = i|z|^2 \)?

If \( z^2 = z\pi \), either \( z = 0 \) or else \( z = \pi \). In other words, \( z \) is real. For \( z^2 = i|z|^2 \), write \( z = re^{i\theta} \) in polar form, and get that either \( r = 0 \) or else \( e^{i\theta} = e^{i(\pi/2-\theta)} \), so \( \theta = \pi/4 + k\pi \), in other words the real and imaginary parts of \( z \) must be equal.

2. Let \( f(z) = z + 1/z \). What is the image of the unit circle under the mapping defined by \( f \)?

The unit circle is \( z = e^{i\theta} \), and \( f(e^{i\theta}) = e^{i\theta} + e^{-i\theta} = 2 \cos \theta \). Since \( \theta \) is real, we get that the image of the unit circle is the real interval \([-2,2]\).

3. On the domain \( \{z = x + iy, 0 \leq x \leq 2\pi, 0 \leq y \leq 2\pi\} \), what is the maximum value of \( |\cos z| \)?

A computation shows that \( |\cos z|^2 = \cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y \). The only critical points of this function on the square in question all have \( y = 0 \), so we need only look on the boundary of the square.

So let \( \phi = |\cos z|^2 \). When \( y = 0 \), \( \phi = \cos^2 x \) so the max is 1. If \( x = 0 \) or \( x = 2\pi \), then \( \phi = \cosh^2 y \), and the max occurs at \( y = 2\pi \) and is \( \cosh^2(2\pi) \). And since \( \sinh^2 2\pi < \cosh^2 2\pi \), \( \phi \) never gets any bigger than this on the line \( y = 2\pi \). Thus the max of \( |\cos z| \) is \( \cosh 2\pi \), which occurs at \( z = 2\pi i, \pi + 2\pi i, 2\pi + 2\pi i \).

4. Let \( u(x, y) = 2x - xy \). Find a function \( v(x, y) \) so that

\[
 f(x + iy) = u(x, y) + iv(x, y)
\]

is a holomorphic function. Express \( f(z) \) in terms of \( z \) alone.

Need \( v_y = u_x = 2 - y \), so \( v = 2y - y^2/2 + f(x) \). Also need \( v_x = -u_y = x \), so \( f(x) = x^2/2 + c \). Thus (up to adding a constant)

\[
 f = 2x - xy + i(x^2/2 - y^2/2 + 2y) = 2z - iz^2/2.
\]

5. Find all the solutions of \( \sin z = \sqrt{3} \).

This is the same as \( q - 1/q = 2i\sqrt{3} \), where \( q = e^{iz} \). Solve the resulting quadratic equation and get \( q = i(\sqrt{3} \pm 2) \).

So

\[
 z = -i \ln q = (\pi/2 + 2k\pi) - i \ln(2 + \sqrt{3})
\]

or

\[
 z = -i \ln q = (-\pi/2 + 2k\pi) - i \ln(2 - \sqrt{3}).
\]

6. Calculate \( \int_\gamma \bar{\zeta} \, dz \), \( \int_\gamma \frac{dz}{\bar{\zeta}} \), where \( \gamma \) is the unit circle, traversed once in the counterclockwise direction.

Let \( z = e^{i\theta} \), note \( \bar{\zeta} = e^{-i\theta} \) and \( dz = ie^{i\theta} d\theta \). Substitute, integrate and get

\[
 \int_\gamma \bar{\zeta} \, dz = 2\pi i
\]

and

\[
 \int_\gamma \frac{dz}{\bar{\zeta}} = 0.
\]
7. Give an example of a (nontrivial) simple closed curve $\gamma$ for which
\[ \int_{\gamma} \frac{dz}{z^2 + z + 1} = 0 \]
and another for which
\[ \int_{\gamma} \frac{dz}{z^2 + z + 1} \neq 0. \]
What is the value of the second integral over your curve?
The roots of $z^2 + z + 1 = 0$ are $z = -1/2 \pm \frac{\sqrt{3}}{2}$, so use partial fractions to get
\[ \frac{1}{z^2 + z + 1} = \frac{1}{2} \left( \frac{1}{z + \frac{1}{2} - \frac{\sqrt{3}}{2}} - \frac{1}{z + \frac{1}{2} + \frac{\sqrt{3}}{2}} \right). \]
So if the curve doesn’t enclose either singularity (or in fact if it encloses both of them) then the integral is 0 (e.g., $|z - 20| = 1$ or $|z| = 10$).
If the curve encloses $-1/2 + i\sqrt{3}/2$ (but not $-1/2 - i\sqrt{3}/2$) then the value of the integral is $2\pi i (1/(i\sqrt{3})) = 2\pi / \sqrt{3}$. For instance $|z + 1/2 - i\sqrt{3}/2| = \sqrt{3}/2$.

8. Calculate
\[ \int_{-\infty}^{\infty} \frac{\cos x}{x^2 + 1} \, dx \]
by applying the Cauchy Integral Formula to
\[ \int_{\gamma} \frac{e^{iz}}{(z + i)(z - i)} \, dz \]
where $\gamma$ is the “standard” semicircular contour of radius $R$ and letting $R$ go to infinity. Be sure to estimate what happens on the circle part carefully.
You can use $f(z) = e^{iz}/(z + i)$ around the standard contour, and get that the integral of $e^{iz}/(z^2 + 1)$ equals $2\pi i f(i) = \pi / e$. If we can show that the integral over the curved part goes to zero, then the improper integral (being the real part of the complex one), is also $\pi / e$.
To show the integral over the curved part goes to zero, let $z = R e^{i\theta}$, and after parametrizing and substituting, we get the integral is less than the integral:
\[ \int_{0}^{\pi} \frac{R}{R^2 - 1} \, d\theta, \]
which certainly goes to zero as $R \to \infty$. 