Math 425 **Dr. DeTurck**
Notes and exercises on Black-Scholes **April 2016**

On Thursday we talked in class about how to derive the Black-Scholes differential equation, which is used in mathematical finance to assign a value to a "financial derivative". The latter is usually the option to buy (a "call" option) or sell (a "put" option) a share of stock at a price specified in advance at some date in the future.

We're only going to discuss what are called "European" options, which are options that can be exercised only at the "expiry date" in the future. There are also "American" options, which can be exercised at any time up to the expiry date. These lead to interesting PDE problems as well (so-called "free boundary problems"), but we'll save that for some later time.

There are lots of assumptions, some realistic and some not, that go into the derivation. For instance, we assume that the risk-free interest rate r (what you could get by putting your money in the bank or in government-backed securities) is a constant, so that if you knew that the value of the stock at time T was going to be S, then the value at time $t < T$ will be $Se^{-r(T-t)}$, the standard present-value computation.

A second assumption is that the return on the stock follows a special kind of random walk called a Wiener process – this is the same process one observes in Brownian motion. Basically, it says that if you know a particle's position at time $t = 0$, then all you can say about its position in the future is that it is a normally-distributed random variable with mean 0 and *variance* proportional to the time t, say $\sigma^2 = 2kt$.

If we remember from stat class that the probability density function for the normal distribution with mean 0 and variance σ^2 is

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-x^2/(2\sigma^2)},$$

then the probability density function for the position of our particle at time t is

$$F(x,t) = \frac{1}{\sqrt{4\pi kt}} e^{-x^2/4kt}$$

for $t > 0$, which we recognize as the fundamental solution of the heat equation.

So we should be on the lookout for something related to the heat equation. There are two complications:

- Since it's the *return* and not the *price* of the stock that changes according to the Wiener process, it's the logarithm of the price that is normally distributed, and we'll have to be alert for an appropriate change of variable
- When we think about our stock options, the only time we actually know their value as a function of the stock price is at the future time when the option is to be exercised, and so we'll have to solve for the current value of the option by working *backwards* in time.

To derive the Black-Scholes equation, we'll indulge in a tiny bit of the theory of stochastic processes, picking off only what we need. If you take a course in stochastic processes sometime, you'll learn this stuff much more rigorously and in greater detail.

Suppose $S(t)$ is the price of the asset (stock) on which our option will be based. The rate of return on the asset is dS/S, and it will have two parts: a deterministic part $\mu\, dt$ that describes in general how the asset's price changes with time – the parameter μ is estimated based on the history of the asset's price. For example, a bank deposit will have return rate $r\, dt$ as noted above. The other part of the return rate will be stochastic, based on the random walk described above, and we'll write it as $\sigma\, dX$, where σ is called the *volatility* of the asset (which is again estimated using past performance) and dX represents the *standard* random Wiener process, so that

$$dX = \phi\sqrt{dt}$$

where ϕ is a standard normal random variable, so that $E[\phi] = 0$ and $E[\phi^2]$=1. Here $E[\cdots]$ is the expectation (or expected value) of a random variable – if the random variable X has probability density function $f(x)$, then

$$E[X] = \int_{-\infty}^{\infty} x f(x)\, dx.$$

so

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f(x)\, dx$$

and it would be the same for any function of X.

Recalling that the variance of a random variable is the expected value of the square of the difference between a random variable and its mean, we know that

$$\text{Var}[X] = E[(X-E[X])^2] = E[X^2]-E[E[X]X]+E[E[X](X-E[X])] = E[X^2]-(E[X])^2,$$

owing to the linearity of expectation, the fact that $E[X]$ is a number, and the expected value of $X - E[X]$ is of course zero.

So we start from

$$\frac{dS}{S} = \sigma\, dX + \mu\, dt$$

and these observations about expection to calculate:

$$E[dS] = E[\sigma S\, dX + \mu S\, dt] = \mu S\, dt$$

because $E[dX] = \sqrt{dt}E[\phi] = 0$, and

$$\text{Var}[dS] = E[(dS)^2] - (E[dS])^2 = E[\sigma^2 S^2 (dX)^2] = \sigma^2 S^2\, dt,$$

because of $E[dX] = 0$ again, and the fact that $E[(dX)^2] = E[\phi^2\, dt] = dt$.

The deep result we need from stochastic calculus is called Itô's Lemma. It describes how a function of a random variable changes in the same way that Taylor's theorem describes how a function of an ordinary deterministic variable changes. To provide some motivation (not to be confused with *proof*), we start with Taylor's theorem as follows:

If $f(S)$ is a smooth function of S, then Taylor's theorem tells us that:

$$df = \frac{df}{dS}\, dS + \frac{1}{2!}\frac{d^2 f}{dS^2}\,(dS)^2 + \cdots$$

where we're writing dS instead of $S - S_0$ and df instead of $f(S) - f(S_0)$ for convenience.

In our stochastic world, we have that $(dX)^2$ is comparable to dt, so if we replace dS by the expression we postulated for dS above, namely

$$dS = \sigma S\, dX + \mu S\, dt,$$

and discard all terms of degree higher than 1 (counting dt as having degree 1 and dX as having degree 1/2), then Taylor's theorem becomes:

$$\begin{aligned} df &= \frac{df}{dS}(\sigma S\, dX + \mu S\, dt) + \frac{1}{2}\sigma^2 S^2 \frac{d^2 f}{dS^2}\, dt + \cdots \\ &= \sigma S \frac{df}{dS}\, dX + \left(\mu S \frac{df}{dS} + \frac{1}{2}\sigma^2 S^2 \frac{d^2 f}{dS^2}\right) dt + \cdots \end{aligned}$$

Dropping the $\cdots$ and thinking of df as an actual (stochastic) differential, this is *Itô's Lemma* for a function f of a random variable S:

$$df = \sigma S \frac{df}{dS}\, dX + \left(\mu S \frac{df}{dS} + \frac{1}{2}\sigma^2 S^2 \frac{d^2 f}{dS^2}\right) dt.$$

We will need a slight extension of Itô's Lemma, to functions that depend on the deterministic time variable t as well as on S. But this extension is easy becuase it just involves adding the linear dt term from the two-variable Taylor series as follows:

$$df = \sigma S \frac{\partial f}{\partial S}\, dX + \left(\mu S \frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + \frac{\partial f}{\partial t}\right) dt.$$

We're almost ready to derive the Black-Scholes equation, but we need a few financial observations concerning stock options. We'll use these to get our initial (actually, "final") data for our PDE problem, as well as at a key juncture in the derivation of the PDE itself. The fundamental observation is that we know what the value C of a (European) call option will be at the moment T it expires as a function of the stock price S at that moment. Namely, since the call option gives us the right to buy a share of the stock at the "strike price" E, the option will be worthless ($C = 0$) if $S < E$ (why pay E when you can buy the stock on the open market for S?), and will be worth the difference $S - E$ between the actual price and the strike price if $S > E$. We'll write this as

$$C(S, T) = (S - E)_+.$$

Likewise, we know the value P of a (European) put option at time T. Since this is the option to sell a share at price E, it's worthless if $S > E$ and worth $E - S$ if $S < E$. We'll write this as

$$P(S,T) = (E - S)_+.$$

Using these expressions, we can construct a collection of items whose value is completely determined at time T. Namely, we buy a share of stock and a put option with strike price E and expiry time T, and "sell short" (sell something we don't own) a call option with the same E and T. At time T, this bundle will be worth:

$$S + P(S,T) - C(S,T) = S + (E - S)_+ - (S - E)_+ = E.$$

And then the standard present value calculation tells us that at any time $t < T$, this bundle will be worth:

$$S + P - C = Ee^{-r(T-t)}.$$

In finance-speak, this equation is called *put-call parity.*

Now we're ready to derive the Black-Scholes equation. We'll do it for a more-or-less arbitrary financial derivative, which could be either a put option, a call option, some combination of these, or some other kind of instrument as long as its value at the expiry time T is known as a function of the stock price S at that time. So we'll derive the equation for $V(S,t)$, the value of the derivative at time t and stock price $S(t)$ (so V would be C as above for a call option and P for a put option).

From Itô's Lemma, we know that the differential dV satisfies:

$$dV = \sigma S \frac{\partial V}{\partial S}\, dX + \left(\mu S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} \right) dt.$$

To get a partial differential equation from this stochastic one, we'll create another derivative that is identical to the original one, except that we'll also sell short k shares of the underlying stock. So the value W of this new derivative is $W = V - kS$, and the stochastic equation for the value W is then

$$dW = \sigma S \left(\frac{\partial V}{\partial S} - k \right) dX + \left(\mu S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} - \mu k S \right) dt.$$

Our objective here is to eliminate the random element, so we want to choose k so that the dX term will disappear from the equation, and only the deterministic dt term will remain. We can do this (to first order) by choosing $k = \frac{\partial V}{\partial S}$, which will make

$$dW = \left(\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} \right) dt.$$

Now, since dW is deterministic, the finance concept of arbitrage (which says basically that if the value of an asset is deterministic, i.e., risk-free, then its value has to increase at the same rate as the prevailing interest rate r), we get that $dW = rW\, dt$. And using the definitions of W and k, we see that

$$r \left(V - \frac{\partial V}{\partial S} S \right) dt = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt.$$

Now we can divide through by dt to get the *Black-Scholes equation*:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0.$$

In this equation, we're looking for $V(S,t)$ and the interest rate r and the stock's volatility σ are "known" constants. It's interesting that the stock's growth rate μ doesn't appear in the equation at all.

Now we (that is, you) need to solve the equation with various "final" conditions at time T. In particular, we need to do this for C and P with the conditions given above.

To derive the solution, the main part of the work is to convert the Black-Scholes equation into the usual heat equation. To do this, you'll have to make three kinds of changes of variable:

- To get the time running in the right direction, you can define a new variable $\tau = T - t$. Then $t = T$ will correspond to $\tau = 0$.
- Since it was $dS/S = d(\log S)$ that satisfied the standard Wiener process that leads to the usual heat equation, it makes sense to define a new variable $x = \log S$ (natural logarithm). This should get rid of the appearances of the independent variable S or x multiplying the various derivatives.
- As we did at the beginning of the course, a substitution of the form $u = e^{\alpha x + \beta\tau}V$ can be used to get rid of unwanted constants and first-order in x terms.

1. With these hints, show that the value of a European call option with strike price E and expiry time T is given by:
$$C(S,t) = SF(A_+) - Ee^{-r(T-t)}F(A_-)$$
where $F(x)$ is the cumulative distribution function for the standard normal distribution:
$$F(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-p^2/2}\,dp,$$
and the constants $A_\pm$ are given by
$$A_\pm = \frac{\log(S/E) + (r \pm \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$
2. Find the value of a European put option in more-or-less the same form.
3. Verify that your formulas satisfy put-call parity.
4. Find the value $B(S,t)$ of a straight bet on the stock price: this is an option that pays \$1 if the stock price S is greater than E at expiry time T, otherwise it pays nothing.
5. Use Maple or some other computer program to draw some graphs of C, P and B for some reasonable choices of r, E, T and σ (if time is measured in years, then reasonable choices for T are between 0.5 and 1.5 and for σ around 0.25).