Math 425 **Dr. DeTurck**
Midterm 1 **March 2, 2010**

There are four problems on this test. You may use your book and your notes during this exam. Do as much of it as you can during the class period, and turn your work in at the end. But take the sheet with the problems home with you, and you may (re)work any problems you like and turn them in on Thursday for additional credit.

1. Solve $u_x - yu_y + 2u = 1$, $u(x, 1) = 0$. In what domain in the plane is your solution determined by the equation (even though the formula you get for u might define a valid function beyond this region)?

   To solve the problem, consider the system of ordinary differential equations:

   $$\frac{dx}{ds} = 1 \qquad \frac{dy}{ds} = -y \qquad \frac{du}{ds} + 2u = 1$$

   together with the initial conditions

   $$x(0) = t \qquad y(0) = 1 \qquad u(0) = 0.$$

   The equations are completely decoupled and the solutions are:

   $$x = s + t \qquad y = e^{-s} \qquad u = \frac{1}{2}(1 - e^{-2s})$$

   From the y equation, $s = -\ln y$. Therefore

   $$u(x, y) = \frac{1}{2}(1 - e^{2\ln y}) = \frac{1}{2}(1 - y^2).$$

   Since $y > 0$ for all s (and t), the solution is determined only in the "upper half plane" (in other words, for points (x, y) with $y > 0$).

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2. Find the general solution $u(x, y)$ of the equation $3u_x + u_{xy} = 1$.

   Let $v = u_x$, then the equation says that $3v + v_y = 1$. This is a linear *ordinary* differential equation with general solution $v = \frac{1}{3} + c_1(x)e^{-3y}$ To find u, we integrate this with respect to x and get

   $$u(x, y) = \frac{x}{3} + C(x)e^{-3y} + K(y)$$

   (where $C(x)$ is the integral of $c_1(x)$).

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3. Let $u(x, t)$ be the temperature in a rod of length L that satisfies the partial differential equation:

   $$u_t = ku_{xx} - ru \qquad \text{for } (x, t) \in (0, L) \times (0, \infty),$$

   where k and r are positive constants – this is related to the heat equation, but assumes that heat radiates out into the air along the rod – together with the initial condition

   $$u(x, 0) = \phi(x)$$

for $x \in [0, L]$, where ϕ satisfies $\phi(0) = \phi(L) = 0$ and $\phi(x) > 0$ for $x \in (0, L)$.

(a) If u also satisfies the Dirichlet boundary conditions

$$u(0,t) = 0, \qquad u(L,t) = 0$$

(so that the ends of the rod are held at temperature 0), show that the total heat energy in the rod at time t, which is given by

$$E(t) = \int_0^L u^2(x,t)\,dx,$$

is a strictly decreasing function of t.

(b) Show that even if u satisfies Neumann boundary conditions

$$u_x(0,t) = 0, \qquad u_x(L,t) = 0$$

(so that the ends of the rod are *insulated*), it is still the case that $E(t)$ as defined above is still a strictly decreasing function of t.

(c) (Extra credit!) Prove that in either (a) or (b), it must be the case that

$$\lim_{t\to\infty} E(t) = 0.$$

(a) To show $E(t)$ is decreasing, calculate its derivative:

$$\frac{dE}{dt} = \int_0^L 2uu_t\,dx = \int_0^L u(ku_{xx} - ru)\,dx$$

Integrate the product uu_{xx} by parts (with $f = u$ and $dg = u_{xx}\,dx$, so $df = u_x\,dx$ and $g = u_x$) and get

$$\frac{dE}{dt} = uu_x\Big|_0^L - \int_0^L 2\Big(k(u_x)^2 + ru^2\Big)\,dx \tag{$*$}$$

Since $u(0,t) = u(L,t) = 0$, the first term is zero. And since $u(x,0) > 0$ and $k > 0$ and $r > 0$, we can conclude that

$$\frac{dE}{dt} = -\int_0^L 2\Big(k(u_x)^2 + ru^2\Big)\,dx < 0$$

and so $E(t)$ is decreasing.

(b) Use the same calculation. Since $u_x(0,t) = u_x(L,t) = 0$, the first term in $(*)$ is zero again, and then the same reasoning shows that E is decreasing because its derivative is negative.

(c) In both (a) and (b) we can conclude that

$$\frac{dE}{dt} \leq -2rE$$

in other words

$$\frac{1}{E}\frac{dE}{dt} \leq -2r.$$

Integrating both sides with respect to t (from 0 to t) gives

$$\ln(E(t)) - \ln(E(0)) \leq -2rt.$$

Now, exponentiating both sides (and multiplying by $E(0)$) gives:

$$E(t) \leq E(0)e^{-2rt}.$$

Therefore $0 \leq E(t) \leq e^{-2rt}$, and since the limits of 0 and e^{-2rt} as $t \to \infty$ are both zero, so is the limit of $E(t)$.

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4. This problem concerns d'Alembert's solution to the initial-value problem for the wave equation $u_{tt} = c^2 u_{xx}$, together with initial conditions

$$u(x,0) = f(x), \qquad u_t(x,0) = g(x).$$

(a) Show that if $f(x)$ and $g(x)$ are periodic functions with period $2L$ (so $f(x+2L) = f(x)$ for all x, and likewise for g), and if

$$\int_{-L}^{L} g(x)\,dx = 0,$$

then $u(x,t)$ is *always* periodic (in x) with period $2L$ (in other words,

$$u(x+2L,t) = u(x,t)$$

for all x and t).

(b) (Continuation of part (a)) With the periodicity assumptions of part (a), show that $u(x,t)$ is also periodic in t. What is its period?

(c) (Separate from parts (a) and (b)) Now suppose that $f(x)$ and $g(x)$, rather than being periodic, actually vanish outside of some finite interval, i.e., $f(x) = 0$ and $g(x) = 0$ for $|x| > R$. Show that

$$\lim_{t\to\infty} u(x,t)$$

is independent of x and give an expression for the limit in terms of f and/or g.

(a) d'Alembert's solution is:

$$u(x,t) = \frac{1}{2}\Big(f(x-ct) + f(x+ct)\Big) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds$$

Therefore

$$u(x+2L,t) = \frac{1}{2}\Big(f(x+2L-ct) + f(x+2L+ct)\Big) + \frac{1}{2c}\int_{x+2L-ct}^{x+2L+ct} g(s)\,ds.$$

By the periodicity of f, we have that $f(x+2L-ct) = f(x-ct)$ and $f(x+2L+ct) = f(x+ct)$ so the first part of the solution is clearly periodic with period $2L$. As for the second, let $r = s - 2L$ so $dr = ds$, and then

$$\int_{x+2L-ct}^{x_2 L+ct} g(s)\,ds = \int_{x-ct}^{x+ct} g(r+2L)\,dr = \int_{x-ct}^{x+ct} g(r)\,dr$$

by the periodicity of g. Therefore

$$\begin{aligned} u(x+2L,t) &= \frac{1}{2}\Big(f(x+2L-ct) + f(x+2L+ct)\Big) + \frac{1}{2c}\int_{x+2L-ct}^{x+2L+ct} g(s)\,ds \\ &= \frac{1}{2}\Big(f(x-ct) + f(x+ct)\Big) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds \\ &= u(x,t). \end{aligned}$$

(b) If we want ct to advance by $2L$, then we need t to advance by $2L/c$. This is a good guess for the period in t:

$$
\begin{aligned}
u\left(x, t+\frac{2L}{c}\right) &= \frac{1}{2}\left[f\left(x-c\left(t+\frac{2L}{c}\right)\right)+f\left(x+c\left(t+\frac{2L}{c}\right)\right)\right]+\frac{1}{2c}\int_{x-c(t+\frac{2L}{c})}^{x+c(t+\frac{2L}{c})} g(s)\,ds \\
&= \frac{1}{2}\left[f\left(x-ct-2L\right)+f\left(x+ct+2L\right)\right]+\frac{1}{2c}\int_{x-ct-2L}^{x+ct+2L} g(s)\,ds \\
&= \frac{1}{2}\left[f\left(x-ct\right)+f\left(x+ct\right)\right]+\frac{1}{2c}\left[\int_{x-ct-2L}^{x-ct} g(s)\,ds+\int_{x-ct}^{x+ct} g(s)\,ds\int_{x+ct}^{x+ct+2L} g(s)\,ds\right] \\
&= \frac{1}{2}\left[f\left(x-ct\right)+f\left(x+ct\right)\right]+\frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds \\
&= u(x,t)
\end{aligned}
$$

using the periodicity of f (to go from the second line to the third), and the periodicity of g together with the fact that the integral of g from $-L$ to L is zero, which together imply that the integral of g over *any* interval of length $2L$ is zero, to go from the third line to the fourth. So u is t-periodic with period L/c.

(c) For any fixed value of x, if we choose t large enough then $x-ct<-R$ and $x+ct>R$ (in other words, choose t so that $t>(R+x)/c$ and $t>(R-x)/c$, noting that the latter is larger if $x<0$), then $f(x+ct)=0$ and $f(x-ct)=0$¿ So the initial position data $f(x)$ will not enter into the limit as $t\to\infty$. Likewise, if we choose t so that $x-ct<R$ and $x+ct>R$, then the integral of g will not change if we make t even bigger, since $g(s)$ is zero when $|s|>R$. Therefore

$$\lim_{t\to\infty} u(x,t)=\frac{1}{2c}\int_{-R}^{R} g(s)\,ds.$$