Math 425 **Dr. DeTurck**
Midterm 1 **March 2, 2010**

There are four problems on this test. You may use your book and your notes during this exam. Do as much of it as you can during the class period, and turn your work in at the end. But take the sheet with the problems home with you, and you may (re)work any problems you like and turn them in on Thursday for additional credit.

1. Solve $u_x - yu_y + 2u = 1$, $u(x, 1) = 0$. In what domain in the plane is your solution determined by the equation (even though the formula you get for u might define a valid function beyond this region)?

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2. Find the general solution $u(x, y)$ of the equation $3u_x + u_{xy} = 1$.

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3. Let $u(x, t)$ be the temperature in a rod of length L that satisfies the partial differential equation:

$$u_t = ku_{xx} - ru \qquad \text{for } (x, t) \in (0, L) \times (0, \infty),$$

where k and r are positive constants – this is related to the heat equation, but assumes that heat radiates out into the air along the rod – together with the initial condition

$$u(x, 0) = \phi(x)$$

for $x \in [0, L]$, where ϕ satisfies $\phi(0) = \phi(L) = 0$ and $\phi(x) > 0$ for $x \in (0, L)$.

(a) If u also satisfies the Dirichlet boundary conditions

$$u(0, t) = 0, \qquad u(L, t) = 0$$

(so that the ends of the rod are held at temperature 0), show that the total heat energy in the rod at time t, which is given by

$$E(t) = \int_0^L u^2(x, t)\, dx,$$

is a strictly decreasing function of t.

(b) Show that even if u satisfies Neumann boundary conditions

$$u_x(0, t) = 0, \qquad u_x(L, t) = 0$$

(so that the ends of the rod are *insulated*), it is still the case that $E(t)$ as defined above is still a strictly decreasing function of t.

(c) (Extra credit!) Prove that in either (a) or (b), it must be the case that

$$\lim_{t\to\infty} E(t) = 0.$$

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4. This problem concerns d'Alembert's solution to the initial-value problem for the wave equation $u_{tt} = c^2 u_{xx}$, together with initial conditions

$$u(x,0) = f(x), \qquad u_t(x,0) = g(x).$$

(a) Show that if $f(x)$ and $g(x)$ are periodic functions with period $2L$ (so $f(x+2L) = f(x)$ for all x, and likewise for g), and if

$$\int_{-L}^{L} g(x)\, dx = 0,$$

then $u(x,t)$ is *always* periodic (in x) with period $2L$ (in other words,

$$u(x+2L,t) = u(x,t)$$

for all x and t).

(b) (Continuation of part (a)) With the periodicity assumptions of part (a), show that $u(x,t)$ is also periodic in t. What is its period?

(c) (Separate from parts (a) and (b)) Now suppose that $f(x)$ and $g(x)$, rather than being periodic, actually vanish outside of some finite interval, i.e., $f(x) = 0$ and $g(x) = 0$ for $|x| > R$. Show that

$$\lim_{t\to\infty} u(x,t)$$

is independent of x and give an expression for the limit in terms of f and/or g.