Math 425 **Dr. DeTurck**
Midterm 1 **February 8, 2007**

1. Solve $u_x + yu_y + u = 0$, $u(0, y) = y$. In what domain in the plane is your solution valid?

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2. Let $u(x, t)$ be the temperature in a rod of length L that satisfies the partial differential equation:

$$u_t = ku_{xx}$$

for $(x, t) \in (0, L) \times (0, \infty)$, where k is a positive constant, together with the initial condition

$$u(x, 0) = \phi(x)$$

for $x \in [0, L]$, where ϕ satisfies $\phi(0) = \phi(L) = 0$ and $\phi(x) > 0$ for $x \in (0, L)$.

(a) If u also satisfies the Neumann boundary conditions

$$u_x(0, t) = 0, \qquad u_x(L, t) = 0,$$

show that the average temperature in the rod at time t, which is given by

$$A(t) = \frac{1}{L} \int_0^L u(x, t)\, dx$$

is a constant (independent of t).

(b) On the other hand, if u satisfies the Dirichlet boundary conditions

$$u(0, t) = 0, \qquad u(L, t) = 0,$$

show that it must be the case the $u(x, t) \geq 0$ for all (x, t) satisfying $0 < x < L$ and $t > 0$.

(c) Still under the assumption that u satisfies the Dirichlet boundary conditions, show that $A(t)$ is a *non-increasing* function of t. (Hint for (a) and (c): Use an argument similar to an energy argument).

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3. (a) Solve the wave equation with friction: $u_{xx} = u_{tt} + 2u_t$ for $0 < x < \pi$ and $t > 0$ with the initial conditions $u(x, 0) = \sin x$, $u_t(x, 0) = 0$, and the boundary conditions $u(0) = u(\pi) = 0$. (Hint: Look for "separated solutions")

(b) If

$$E(t) = \frac{1}{2} \int_0^\pi u_t^2 + u_x^2\; dx,$$

show that

$$\lim_{t \to \infty} E(t) = 0.$$

(Hint: To do this, you can calculate $E(t)$ explicitly).

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4. Find as general a solution $u(x, y, z)$ as you can to the third-order equation

$$u_{xyz} = 0$$