Math 425 / AMCS 525 **Dr. DeTurck**
Final Exam **May 4, 2016**

You may use your book and notes on this exam. Show your work in the exam book. Work only the problems that correspond to the section that you prepared.

Section 9.4 (The diffusion equation in 3 dimensions)

1. (a) For the Dirichlet problem for the heat equation:

$$u_t = \Delta u \quad \text{on } D \subset R^3$$

$$u(\overline{\mathbf{x}}, t) = 0 \quad \text{for } \overline{\mathbf{x}} \in \partial D, \text{ all } t > 0$$

$$u(\overline{\mathbf{x}}, 0) = 0 \quad \text{for } \overline{\mathbf{x}} \in D,$$

show that the only solution is $u(\overline{\mathbf{x}}, t) = 0$. (Use the energy method, let $E(t) = \iiint_D (u(\overline{\mathbf{x}}, t))^2 \, d\overline{\mathbf{x}}$ and adapt the proof from section 2.3).

(b) Use the result from part (a) to prove *uniqueness* for the problem:

$$u_t = \Delta u + f(\overline{\mathbf{x}}, t) \quad \text{on } D \subset R^3$$

$$u(\overline{\mathbf{x}}, t) = g(\overline{\mathbf{x}}) \quad \text{for } \overline{\mathbf{x}} \in \partial D, \text{ all } t > 0$$

$$u(\overline{\mathbf{x}}, 0) = h(\overline{\mathbf{x}}) \quad \text{for } \overline{\mathbf{x}} \in D,$$

(a) Calculate

$$\begin{aligned} \frac{dE}{dt} &= \frac{d}{dt} \iiint_D u^2 \, d\overline{\mathbf{x}} = 2 \iiint_D u u_t \, d\overline{\mathbf{x}} = 2 \iiint x \Delta u \, d\overline{\mathbf{x}} \\ &= \iiint_D u \nabla \cdot \nabla u \, d\overline{\mathbf{x}} = \iiint_D \nabla \cdot (u \nabla u) - u \Delta u \, d\overline{\mathbf{x}} \\ &= \iint_{\partial D} u \nabla u \cdot n \, d\text{area} = 0 \end{aligned}$$

because $\Delta u = 0$ and $u = 0$ on ∂D. So the energy is constant in time, and because $u(\overline{\mathbf{x}}, 0) = 0$ we have $E(0) = 0$. Therefore $E(t) = 0$ for all t and so $u(\overline{\mathbf{x}}, t) = 0$ for all t.

(b) If there are two solutions u_1 and u_2 of this problem, then their difference $v(\overline{\mathbf{x}}, t) = u_1(\overline{\mathbf{x}}, t) - u_2(\overline{\mathbf{x}}, t)$ satisfies the homogenous problem of part (a) and is therefore zero. So we conclude that $u_1 = u_2$, proving uniqueness.

2. On all of R^3, solve the problem

$$u_t = \Delta u \quad \text{on all of } R^3$$

$$u(\overline{\mathbf{x}}, 0) = f(\overline{\mathbf{x}}) \quad \text{for } \overline{\mathbf{x}} \in R^3,$$

where $f(x,y,z) = 1$ for (x,y,z) inside the box where $-1 < x < 1$, $-2 < y < 2$ and $-3 < z < 3$, and $f(x,y,z) = 0$ otherwise. Write your answer in terms of the error function (Erf) as in section 2.4. (Use the form of the fundamental solution given in equation (7) on page 250. It will help to write $f(x,y,z)$ as a product of functions of one variable, and then the solution should decompose the same way.)

First, note that $f(x,y,z) = P(x)Q(y)R(z)$, where

$$P(x) = \begin{cases} 1 & \text{for } -1 < x < 1 \\ 0 & \text{otherwise} \end{cases} \qquad Q(y) = \begin{cases} 1 & \text{for } -2 < y < 2 \\ 0 & \text{otherwise} \end{cases} \qquad R(z) = \begin{cases} 1 & \text{for } -3 < z < 3 \\ 0 & \text{otherwise} \end{cases}$$

Therefore

$$\begin{aligned} u(x,y,z,t) &= \frac{1}{(4k\pi t)^{3/2}} \iiint_{\mathbb{R}^3} e^{-[(x-x')^2+(y-y')^2+(z-z')^2]/4kt} f(x',y',z')\,dx'\,dy'\,dz' \\ &= \left(\frac{1}{\sqrt{4k\pi t}} \int_{-\infty}^{\infty} e^{-(x-x')^2/4kt} P(x')\,dx' \right) \times \left(\frac{1}{\sqrt{4k\pi t}} \int_{-\infty}^{\infty} e^{-(y-y')^2/4kt} Q(y')\,dy' \right) \\ &\qquad \times \left(\frac{1}{\sqrt{4k\pi t}} \int_{-\infty}^{\infty} e^{-(z-z')^2/4kt} R(z')\,dz' \right) \\ &= \left(\frac{1}{\sqrt{4k\pi t}} \int_{-1}^{1} e^{-(x-x')^2/4kt}\,dx' \right) \left(\frac{1}{\sqrt{4k\pi t}} \int_{-2}^{2} e^{-(y-y')^2/4kt}\,dy' \right) \left(\frac{1}{\sqrt{4k\pi t}} \int_{-3}^{3} e^{-(z-z')^2/4kt}\,dz' \right) \end{aligned}$$

So we need to evaluate

$$I_h(x) = \frac{1}{\sqrt{4\mathbf{k}\pi t}} \int_{-h}^{h} e^{-(x-y)^2/4kt}\,dy$$

Make the standard substitution $p = (x-y)/\sqrt{4kt}$ so that $dp = -dy/\sqrt{4kt}$ and obtain

$$\begin{aligned} I_h(x) &= \frac{1}{\sqrt{\pi}} \int_{(x-h)/\sqrt{4kt}}^{(x+h)/\sqrt{4kt}} e^{-p^2}\,dp \\ &= \frac{1}{\sqrt{\pi}} \left(\int_0^{(x+h)/\sqrt{4kt}} e^{-p^2}\,dp - \int_0^{(x-h)/\sqrt{4kt}} e^{-p^2}\,dp \right) \\ &= \frac{1}{2} \left(\mathrm{Erf}\left(\frac{x+h}{\sqrt{4kt}} \right) - \mathrm{Erf}\left(\frac{x-h}{\sqrt{4kt}} \right) \right) \end{aligned}$$

Therefore the solution of our initial-value problem is

$$\begin{aligned} u(x,y,z,t) &= I_1(x) I_2(y) I_3(z) \\ &= \frac{1}{8} \left(\mathrm{Erf}\left(\frac{x+1}{\sqrt{4kt}} \right) - \mathrm{Erf}\left(\frac{x-1}{\sqrt{4kt}} \right) \right) \left(\mathrm{Erf}\left(\frac{y+2}{\sqrt{4kt}} \right) - \mathrm{Erf}\left(\frac{y-2}{\sqrt{4kt}} \right) \right) \left(\mathrm{Erf}\left(\frac{z+3}{\sqrt{4kt}} \right) - \mathrm{Erf}\left(\frac{z-3}{\sqrt{4kt}} \right) \right) \end{aligned}$$

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Section 12.3 (Fourier transforms)

1. Use the Fourier transform (in x) to solve the initial-value problem for the first-order equation

$$u_t = 2u_x + 3u \quad u(x,0) = f(x).$$

Take the Fourier transform of the equation and the data to get

$$\widehat{u}_t = (2i\omega + 3)\widehat{u} \qquad \widehat{u}(\omega, 0) = \widehat{f}(\omega).$$

The solution of this ordinary differential equation for $\widehat{u}$ as a function of t is

$$\widehat{u}(\omega, t) = \widehat{f}(\omega)e^{(3+2i\omega)t}.$$

Now take the inverse Fourier transform of both sides, factor out e^{3t} and use the shifting formula $\mathcal{F}\left[f(x-a)\right] = e^{-ia\omega}\widehat{f}(\omega)$ to get

$$u(x,t) = e^{3t}\mathcal{F}^{-1}\left[e^{2it\omega}\widehat{f}(\omega)\right] = e^{3t}f(x+2t).$$

---

2. Define $C_a(x) = \dfrac{a}{\pi(x^2+a^2)}$.

(a) If $a > 0$, show that $C_a(x)$ is a probability density function (that is, it is positive and $\displaystyle\int_{-\infty}^{\infty} C_a(x)\,dx = 1$). $C_a(x)$ is called a *Cauchy distribution.*
(Hint: You can use the Fourier transform to do this because $e^0 = 1$)

(b) The probability density function (pdf) of the sum of two random variables is the convolution of the pdfs of the two random variables. If A is a random variable with pdf $C_a(x)$ and B is a random variable with pdf $C_b(x)$ for $a > 0$ and $b > 0$, calculate the pdf of the random variable $A+B$. (Again, using the Fourier transform will help avoid doing unpleasant integrals.)

(a) We know that $\mathcal{F}\left[\dfrac{1}{a^2+x^2}\right] = \dfrac{\pi}{a}e^{-a|\omega|}$. So

$$\widehat{C_a}(\omega) = \mathcal{F}\left[\frac{a}{\pi}\,\frac{1}{a^2+x^2}\right] = e^{-a|\omega|}.$$

Also,

$$\widehat{C_a}(0) = \int_{-\infty}^{\infty} C_a(x)e^{-i0x}\,dx = \int_{-\infty}^{\infty} C_a(x)\,dx.$$

So

$$\int_{-\infty}^{\infty} C_a(x)\,dx = \widehat{C_a}(0) = 1$$

and $C_a(x)$ is clearly positive for all x provided $a > 0$, so $C_a(x)$ is a pdf.

(b) The pdf of $A + B$ is

$$C_a * C_b = \mathcal{F}^{-1}\left[\widehat{C_a}\,\widehat{C_b}\right] = \mathcal{F}^{-1}\left[e^{-a|\omega|}e^{-b|\omega|}\right] = \mathcal{F}^{-1}\left[e^{-(a+b)|\omega|}\right] = C_{a+b}(x).$$

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Sections 10.3 and 10.6 (Spherical coordinates and Legendre)

1. Find the steady-state temperature $u(\rho, \varphi, \theta)$ in the solid ball of radius 2 if the surface temperature is given in polar coordinates by

$$u(2, \varphi, \theta) = \sin^2 \varphi$$

for $0 < \varphi < \pi$ in spherical coordinates (here, φ is the elevation angle from the xy-plane and θ is the azimuthal angle).

We need to find the solution of Laplace's equation $\nabla^2 u = 0$ with the given boundary values. In spherical coordinates,

$$\nabla^2 u = \frac{\partial^2 u}{\partial \rho^2} + \frac{2}{\rho}\frac{\partial u}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial \varphi^2} + \frac{\cot\varphi}{\rho^2}\frac{\partial u}{\partial \varphi} + \frac{1}{\rho^2 \sin^2\varphi}\frac{\partial^2 \rho}{\partial \theta^2}.$$

Since the boundary data is independent of θ, the solution will be as well, so the Laplacian reduces to

$$\nabla^2 u = \frac{\partial^2 u}{\partial \rho^2} + \frac{2}{\rho}\frac{\partial u}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial \varphi^2} + \frac{\cot\varphi}{\rho^2}\frac{\partial u}{\partial \varphi}.$$

We separate variables as $u(\rho, \varphi) = R(\rho)\Phi(\varphi)$ and get

$$\frac{\rho^2 R'' + 2\rho R'}{R} = -\frac{\Phi'' + \cot\varphi\,\Phi'}{\Phi} = \lambda$$

so we have

$$\rho^2 R'' + 2\rho R' - \lambda R = 0 \qquad \text{and} \qquad \sin\varphi\,\Phi'' + \cos\varphi\,\Phi' + \lambda \sin\varphi\,\Phi = 0$$

where we need $\Phi(\varphi)$ to be bounded when $\varphi = 0$ and $\varphi = \pi$. Make the substitution $x = \cos\varphi$ in the second equation and it becomes

$$(1 - x^2)\frac{d^2\Phi}{dx^2} - 2x\frac{d\Phi}{dx} + \lambda\Phi = 0$$

for $-1 < x < 1$, which is Legendre's equation. The only solutions of this which are bounded for $x = -1$ and $x = 1$ are the Legendre polynomials $P_n(x)$ which means that $\lambda = n(n+1)$ for $n = 0, 1, 2, \ldots$. So the eigenvalues and eigenfunctions of the second separated equation are

$$\lambda = n(n+1), \qquad \Phi(\varphi) = P_n(\cos\varphi).$$

Next, we consider the solutions of

$$\rho^2 R'' + 2\rho R' - n(n+1)R = 0$$

which is a Cauchy-Euler equation. As usual, we guess $R = \rho^a$ and see that $a(a-1)+2a-n(n+1) = a(a+1)-n(n+1) = 0$ so $a = n$ or $a = -(n+1)$. To have $R(\rho)$ bounded at $\rho = 0$ we use only $a = n$ and so we conclude that

$$u(\rho,\varphi) = \sum_{n=1}^{\infty} a_n \rho^n P_n(\cos\varphi).$$

When $\rho = 2$ we need

$$\sin^2\varphi = \sum_{n=1}^{\infty} 2^n a_n P_n(\cos\varphi).$$

Now $P_0(\cos\varphi) = 1$, $P_1(\cos\varphi) = \cos\varphi$ and $P_2(\cos\varphi) = \frac{1}{2}(3\cos^2\varphi - 1)$. Thus

$$\sin^2\varphi = 1 - \cos^2\varphi = \tfrac{2}{3}P_0(\cos\varphi) - \tfrac{2}{3}P_2(\cos\varphi)$$

so we need $a_0 = \frac{2}{3}$, $a_2 = -\frac{1}{6}$ and all the other $a_n = 0$. We conclude that

$$u(\rho,\varphi) = \tfrac{2}{3}P_0(\cos\varphi) - \tfrac{1}{6}\rho^2 P_2(\cos\varphi) = \tfrac{2}{3} - \tfrac{1}{12}\rho^2(3\cos^2\varphi - 1) = \tfrac{2}{3} + \tfrac{1}{12}\rho^2 - \tfrac{1}{4}\rho^2\cos^2\varphi$$

(Note, in rectangular coordinates, $u = \frac{2}{3} + \frac{1}{12}(x^2 + y^2 - 2z^2)$).

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Section 13.1 (Electromagnetism)

1. Let D be any domain in R^3, and let $\mathbf{V}$ be a vector field on D. Also, let $f = \nabla \cdot \mathbf{V}$ (the divergence of $\mathbf{V}$) defined on D and let $g = \mathbf{V} \cdot \mathbf{n}$ be the (outward pointing) normal component of $\mathbf{V}$ defined on the boundary ∂D.

(a) Explain why

$$\iiint_D f \; d\text{vol} = \iint_{\partial D} g \; d\text{area}$$

(b) Now let u be a solution of the Poisson equation $\Delta u = f$ on D with Neumann boundary values (normal derivative at the boundary) $\dfrac{\partial u}{\partial \mathbf{n}} = g$. Then let $\mathbf{V}_2 = \nabla u$ (the gradient of u) and let $\mathbf{V}_1 = \mathbf{V} - \mathbf{V}_2$. Prove that $\nabla \cdot \mathbf{V}_1 = 0$ (i.e., $\mathbf{V}_1$ is a divergence-free vector field) and the $\mathbf{V}_1 \cdot \mathbf{n} = 0$ on the boundary of D.

(c) Prove that $\mathbf{V}_1$ and $\mathbf{V}_2$ are orthogonal in the L^2 sense, in other words:

$$\langle \mathbf{V}_1, \mathbf{V}_2 \rangle = \iiint_D \mathbf{V}_1 \cdot \mathbf{V}_2 \, d\text{vol} = 0$$

(d) Explain why this means that every vector field on D can be expressed as the sum of a divergence-free vector field and the gradient of a function on D. Is this decomposition unique (i.e., is there any leeway at all in choosing $\mathbf{V}_1$ and $\mathbf{V}_2$ in part (b))? Why or why not?

(a) Two words: "divergence theorem"

(b) First, we have

$$\nabla \cdot \mathbf{V}_1 = \nabla \cdot (\mathbf{V} - \mathbf{V}_2) = \nabla \cdot \mathbf{V} - \nabla \cdot \mathbf{V}_2 = f - \nabla \cdot \nabla u = f - \Delta u = f - f = 0$$

Next, on the boundary of D, we have

$$\mathbf{V}_1 \cdot \mathbf{n} = (\mathbf{V} - \mathbf{V}_2) \cdot \mathbf{n} = \mathbf{V} \cdot \mathbf{n} - \mathbf{V}_2 \cdot \mathbf{n} = g - (\nabla u) \cdot \mathbf{n} = g - \frac{\partial u}{\partial \mathbf{n}} = g - g = 0.$$

(c) We have

$$\langle \mathbf{V}_1, \mathbf{V}_2 \rangle = \langle \mathbf{V} - \mathbf{V}_2, \mathbf{V}_2 \rangle = \iiint_D \mathbf{V}_1 \cdot \nabla u \, d\text{vol}$$

But we have the product rule $\nabla \cdot (u\mathbf{V}_1) = \mathbf{V}_1 \cdot \nabla u + u \nabla \cdot \mathbf{V}_1$. Since $\nabla \cdot \mathbf{V}_1 = 0$ from part (b), we can replace $\mathbf{V}_1 \cdot \nabla u$ by $\nabla \cdot (u\mathbf{V}_1)$ and so

$$\langle \mathbf{V}_1, \mathbf{V}_2 \rangle = \iiint_D \nabla \cdot (u\mathbf{V}_1) \, d\text{vol} = \iint_{\partial D} u \mathbf{V}_1 \cdot \mathbf{n} \, d\text{area} = 0$$

since $\mathbf{V}_1 \cdot \mathbf{n} = 0$ on ∂D by part (b) again.

(d) In part (b), we started with an arbitrary vector field $\mathbf{V}$ and wrote it as $\mathbf{V}_1 + \mathbf{V}_2$, where $\mathbf{V}_1$ is divergence-free and $\mathbf{V}_2 = \nabla u$, which is what the problem is asking. The decomposition is not unique, since we could add the gradient of any harmonic function h to $\mathbf{V}_2$ (so $\mathbf{V}_2 = \nabla(u + h)$ with u as given in part (b)) and the proof that $\nabla \cdot \mathbf{V}_1 = 0$ would still be valid. But $\mathbf{V}_1 \cdot \mathbf{n}$ would not be zero on the boundary in this case, so the decomposition would not necessarily be orthogonal.

2. Write down any (nonconstant - it must depend on t as well as at least one of x, y and/or z) solution of the vacuum Maxwell's equations (1) on page 358.

There are lots of ways to do this. We need at least one of $\mathbf{E}^0$ or $\mathbf{B}^0$ to have a non-zero curl. So let

$$\mathbf{E}(\mathbf{x}, 0) = \mathbf{E}^0 = [0, x, 0] \qquad \text{and} \qquad \mathbf{B}(\mathbf{x}, 0) = \mathbf{B}^0 = [0, 0, 0]$$

Then we need $\mathbf{E}$ and $\mathbf{B}$ to solve the wave equation ($\mathbf{E}_{tt} = c^2 \Delta \mathbf{E}$ and $\mathbf{B}_{tt} = c^2 \Delta \mathbf{B}$) with initial conditions

$$\mathbf{E}(\mathbf{x}, 0) = \mathbf{E}^0 = [0, x, 0] \qquad \text{and} \qquad \frac{\partial \mathbf{E}}{\partial t}(\mathbf{x}, 0) = c \nabla \times \mathbf{B}^0 = [0, 0, 0]$$

and

$$\mathbf{B}(\mathbf{x}, 0) = \mathbf{B}^0 = [0, 0, 0] \qquad \text{and} \qquad \frac{\partial \mathbf{B}}{\partial t}(\mathbf{x}, 0) = -c \nabla \times \mathbf{E}^0 = [-c, 0, 0].$$

It's pretty easy to see that

$$\mathbf{E}(\mathbf{x}, t) = [0, x, 0] \qquad \text{and} \qquad \mathbf{B}(\mathbf{x}, t) = [-ct, 0, 0]$$

satisfy this, and so satisfy the Maxwell equations.

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Section 14.1 (Shock waves)

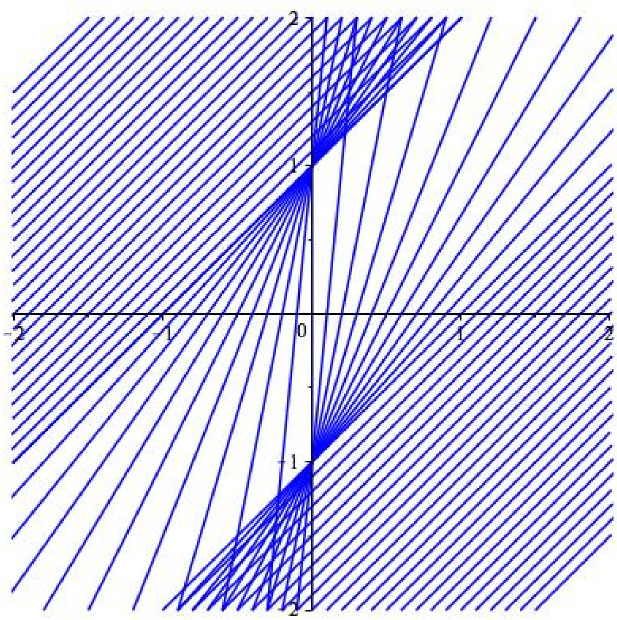
Consider the initial-value problem for Burger's equation:

$$u_t + uu_x = 0$$

with the initial data

$$u(x,0) = \begin{cases} 1 & \text{for } |x| > 1 \\ |x| & \text{for } |x| \le 1 \end{cases}$$

1. Shetch the characteristics in the (x,t) plane.



2. Find the "classical" (continuous) part of the solution, and determine the time of shock formation. On what part of the plane is the the continuous part of the solution valid?

From the picture, you can see that the solution will be valid wherever the characteristics do not cross, i.e., for $x > 0$ and $t < x + 1$ as well as for $x < -$ and $t > x - 1$.
The continuous part of the solution is given in four parts:

$$u(x,t) = \begin{cases} 1 & \text{if } x < 0 \text{ and } t > x - 1 \\ \dfrac{x}{t-1} & \text{if } x < 0 \text{ and } x - 1 < t < x + 1 \\ \dfrac{x}{t+1} & \text{if } x > 0 \text{ and } x - 1 < t < x + 1 \\ 1 & \text{if } x > 0 \text{ and } t < x + 1 \end{cases}$$

So going forward in time from $t = 0$, the first singularity occurs when $t = 1$ (at $x = 0$), and going backward in time the first singularity occurs at $t = -1$ (at $x = 0$).

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Sections 9.4 and 9.5 (Schrödinger's equation, Hermite functions and H_2)

1. The purpose of this problem is to find the norm (squared) of $h_n(x) = e^{-x^2/2}H_n(x)$, i.e.,

$$\langle h_n, h_n\rangle = \int_{-\infty}^{\infty} (h_n(x))^2\,dx = \int_{-\infty}^{\infty} e^{-x^2}(H_n(x))^2\,dx$$

(a) Prove that for all $n \geq 1$,

$$h_{n+1}(x) - 2xh_n(x) + 2nh_{n-1}(x) = 0$$

(b) Use the identity in (a) multiplied by $h_{n-1}(x)$ subtracted from the identity with n replaced by $n-1$ and multiplied by $h_n(x)$ to prove that, for $n \geq 2$,

$$h_n^2(x) + 2(n-1)h_n(x)h_{n-2}(x) - h_{n+1}(x)h_{n-1}(x) - 2nh_{n-1}^2(x) = 0$$

(c) Integrate from $-\infty$ to ∞ (don't forget about the orthogonality of the h_n's that you proved in the homework assignment!) and obtain for $n \geq 2$

$$\langle h_n, h_n\rangle = 2n\,\langle h_{n-1}, h_{n-1}\rangle\,.$$

Also prove that this is true for $n = 1$.

(d) Work this all the way down to $n = 0$ — what is $\|h_n\|^2$?

(a) From the Fourier transform homework, we have the identities

$$xh_n + h_n' = 2nh_{n-1} \qquad \text{and} \qquad xh_n - h_n' = h_{n+1}$$

Add these together and get

$$2xh_n = 2nh_{n-1} + h_{n+1}$$

which can be rearranged to get

$$h_{n+1}(x) - 2xh_n(x) + 2nh_{n-1}(x) = 0$$

(b) Replacing n by $n-1$ gives us that for all $n \geq 2$,

$$h_n(x) - 2xh_{n-1}(x) + 2(n-1)h_{n-2}(x) = 0$$

Multiply this by h_n to get (for $n \geq 2$):

$$h_n^2 - 2xh_nh_{n-1} + 2(n-1)h_nh_{n-2} = 0$$

and multiplying the original identity by h_{n-1} gives:

$$h_{n+1}h_{n-1} - 2xh_nh_{n-1} + 2nh_{n-1}^2 = 0$$

Subtract these to get (for $n \geq 2$)

$$h_n^2 + 2(n-1)h_nh_{n-2} - h_{n+1}h_{n-1} - 2nh_{n-1}^2 = 0$$

(c) If we integrate this last identity from $-\infty$ to ∞, this becomes

$$\langle h_n, h_n \rangle + 2(n-1)\langle h_n, h_{n-2} \rangle - \langle h_{n+1}, h_{n-1} \rangle - 2n \langle h_{n-1}, h_{n-1} \rangle$$

But from the homework we know that h_n is orthogonal to h_m if $n \neq m$, so the two middle terms are zero and we get

$$\langle h_n, h_n \rangle = 2n \langle h_{n-1}, h_{n-1} \rangle .$$

for $n \geq 2$. For $n = 1$ this identity says that $\langle h_1, h_1 \rangle = 2 \langle h_0, h_0 \rangle$, or

$$\int_{-\infty}^{\infty} 4x^2 e^{-x^2}\, dx = 2 \int_{-\infty}^{\infty} e^{-x^2}\, dx$$

But integrating the left integral by parts (with $u = 2x$ and $dv = 2xe^{-x^2}\, dx$, so $du = 2\, dx$ and $v = -e^{-x^2}$) shows that

$$\int_{-\infty}^{\infty} 4x^2 e^{-x^2}\, dx = \left. -2xe^{-x^2} \right|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} 2e^{-x^2}\, dx = 2\int_{-\infty}^{\infty} e^{-x^2}\, dx$$

so we have $\langle h_1, h_1 \rangle = 2 \langle h_0, h_0 \rangle$

(d) By induction, we have that $\|h_n\|^2 = \langle h_n, h_n \rangle = 2^n n! \langle h_0, h_0 \rangle = 2^n n! \sqrt{\pi}$.

---

2. The Hermite polynomials and functions arise in quantum mechanics in the solution of the equation of the one-dimensional quantum-mechanical harmonic oscillator:

$$-iu_t = u_{xx} - x^2 u$$

Separated solutions turn out to be

$$u_k(x,t) = e^{-i(2k+1)t} h_k(x)$$

as reported in the textbook on page 253. Solve the following initial-value problem:

$$-iu_t = u_{xx} - x^2 u \qquad u(x,0) = (24x^3 - 8x^2)e^{-x^2/2}$$

Since we know the separated solutions, this reduces to finding coefficients $a_0, \dots a_3$ so that

$$24x^3 - 8x^2 = a_0 H_0(x) + a_1 H_1(x) + a_2 H_2(x) + a_3 H_3(x).$$

It's easy to see that we need $a_3 = 3$, from which it will follow that $a_1 = 18$. Likewise, we need $a_2 = -2$ from which we get $a_1 = -4$. Therefore the solution is

$$u(x,t) = 3e^{-7it} h_3(x) - 2e^{-5it} h_2(x) + 18e^{-3it} h_1(x) - 4e^{-it} h_0(x).$$

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Section 14.3 (Calculus of Variations)

1. Find the stationary function of

$$\int_0^4 [xy' - (y')^2]\, dx$$

subject to the boundary conditions $y(0) = 0$ and $y(4) = 3$.

We have $F(x, y, p) = xp - p^2$ so the Euler-Lagrange equations $F_u = (F_p)_x$ become

$$0 = \frac{d}{dx}(x - 2y') = 1 - 2y''$$

so we need $y'' = \frac{1}{2}$ or $y = \frac{1}{4}x^2 + ax + b$. To have $y(0) = 0$ we need $b = 0$ and then for $y(4) = 3$ we need $a = -\frac{1}{4}$, so the solution is

$$y(x) = \frac{1}{4}(x^2 - x).$$

2. Find the stationary functions for the integral

$$\int_0^1 y^2 - (y')^2\, dx$$

This time, $F(x, y, p) = y^2 - p^2$ so the Euler-Lagrange equation becomes

$$2y = -\frac{d}{dx}(2y') = -2y''.$$

So we need $y'' = -y$ and we have $y = a\cos x + b\sin x$.

3. (a) Let $F = F(x, y, y')$, where $y' = dy/dx$. Prove the identity:

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}y' - F\right) = y'\left[\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y}\right] - \frac{\partial F}{\partial x}.$$

Conclude that, if the function F does not explicitly depend on x then the Euler-Lagrange equation can be integrated to

$$\frac{\partial F}{\partial y'}\,y' - F = c_1$$

where c_1 is a constant.

(b) Use this to find "minimal" (i.e., stationary) surfaces of revolution obtained by rotating the curve $y = y(x)$ around the x-axis. (Recall that the surface area of the surface of revolution is

$$S = \int_a^b 2\pi y\sqrt{1 + y'^2}\,dx$$

so $F(x, y, p) = 2\pi y\sqrt{1 + p^2}$ in this case.) *Hint*: Persevere with the algebra; at some point you will need to make the substitution $y = c_1 \cosh z$ in the integral $\displaystyle\int \frac{dy}{\sqrt{y^2 - c_1^2}}$.

(a) We have

$$\begin{aligned}\frac{d}{dx}\left(\frac{\partial F}{\partial y'}y' - F\right) &= y'\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + \frac{\partial F}{\partial y'}y'' - \frac{\partial F}{\partial x} - \frac{\partial F}{\partial y}\,y' - \frac{\partial F}{\partial y'}\,y'' \\ &= y'\left[\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y}\right] - \frac{\partial F}{\partial x}.\end{aligned}$$

If we assume that $\dfrac{\partial F}{\partial x} = 0$ and use that the Euler-Lagrange equation is precisely that the expression in square brackets is 0, we can conclude that

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}y' - F\right) = 0$$

and integrating this with respect to x immediately gives that

$$\frac{\partial F}{\partial y'}y' - F = c_1$$

(b) Since F does not depend on x, we can use part (a) and write (after dividing out the pesky 2π)

$$\frac{yy'^2}{\sqrt{1+y'^2}} - y\sqrt{1+y'^2} = c_1$$

Multiply by $\sqrt{1+y'^2}$ and square bothe sides to get

$$y^2 = c_1^2(1+y'^2)$$

or

$$c_1 y' = \sqrt{y^2 - c_1^2}$$

This is a separable ODE with solution

$$x = c_1 \int \frac{dy}{\sqrt{y^2 - c_1^2}}$$

Let $y = c_1 \cosh z$, so $dy = c_1 \sinh z$ and $y^2 - c_1^2 = c_1^2 \sinh^2 z$ and get

$$x = c_1 \int dz = c_1 \operatorname{arccosh} \frac{y}{c_1} + c_2$$

or

$$y = c_1 \cosh\left(\frac{x - c_2}{c_1}\right).$$