Math 425 / AMCS 525 **Dr. DeTurck**
Assignment 1 solutions **Due Tuesday, January 20, 2016**

1. Find the general solution of $x'(t) + x^2 \sin t = 0$.

This is not a linear equation because of the x^2, but it is separable. Rewrite it as

$$\frac{dx}{dx} = -x^2 \sin t$$

and then separate:

$$\frac{dx}{x^2} = -\sin t \, dt$$

and integrate:

$$\int \frac{dx}{x^2} = \int \sin t \, dt$$

$$-\frac{1}{x} = \cos t - C$$

$$x = \frac{1}{C - \cos t}$$

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2. Solve the initial-value problem $x'(t) + x(t) \cos t = 0, \quad x(\pi) = 100$.

The differential equation is both separable and linear, so we'll solve it as a linear equation this time; it is of the form $x' + p(t)x = q(t)$ where $p(t) = \cos t$ and $q(t) = 0$. The general solution is:

$$x = e^{-\int p} \int q e^{\int p} = e^{-\int \cos t \, dt} \int 0 \, dt = Ce^{-\sin t}$$

(recall that in this method you must only add a constant of integration to the "big" integral). Since $x = 100$ when $t = \pi$, we have

$$100 = Ce^{-\sin \pi} = Ce^0 = C$$

so $C = 100$ and the solution of the initial-value problem is

$$x = 100e^{-\sin t}.$$

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3. Find the general solution: $2y'' + 5y' + 2y = 0$.

We need the roots of the polynomial $2r^2 + 5r + 2 = (2r+1)(r+2)$, which are $r = -2$ and $r = -\frac{1}{2}$. So the general solution is

$$y = c_1 e^{-2t} + c_2 e^{-\frac{1}{2}t}.$$

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4. Find the solution of the initial-value problem: $5y'' + 8y' + 5y = 0, \quad y(0) = 1,\ y'(0) = 0.$

This time we need the roots of the polynomial $5r^2 + 8r + 5$, which doesn't factor in an obvious way. So:

$$r = \frac{-8 \pm \sqrt{64 - 100}}{10} = \frac{-8 \pm 6}{10} = -\frac{4}{5} \pm \frac{3}{5}i.$$

So the general solution is

$$y = e^{-\frac{4}{5}t}\left(c_1 \cos \tfrac{3}{5}t + c_2 \sin \tfrac{3}{5}t\right).$$

And since we'll need it, we calculate:

$$\begin{aligned} y' &= -\tfrac{4}{5}e^{-\frac{4}{5}t}\left(c_1 \cos \tfrac{3}{5}t + c_2 \sin \tfrac{3}{5}t\right) + e^{-\frac{4}{5}t}\left(-\tfrac{3}{5}c_1 \sin \tfrac{3}{5}t + \tfrac{3}{5}c_2 \cos \tfrac{3}{5}t\right) \\ &= e^{-\frac{4}{5}t}\Big(\left(\tfrac{3}{5}c_2 - \tfrac{4}{5}c_1\right)\cos \tfrac{3}{5}t + \left(\tfrac{3}{5}c_2 + \tfrac{4}{5}c_1\right)\sin \tfrac{3}{5}t\Big). \end{aligned}$$

Since $e^0 = 1$, $\cos 0 = 1$ and $\sin 0 = 0$, it's easy to see that $y(0) = 1$ implies that $c_1 = 1$. Next, $y'(0) = 0$ becomes $\frac{3}{5}c_2 - \frac{4}{5} = 0$, so $c_2 = \frac{4}{3}$, and the solution of the initial-value problem is

$$y = e^{-\frac{4}{5}t}\Big(\cos \tfrac{3}{5}t + \tfrac{4}{3}\sin \tfrac{3}{5}t\Big).$$

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5. Solve the following system of differential equations for $x(t)$ and $y(t)$:

$$x'(t) = x(t) - 4y(t), \quad y'(t) = x(t) + y(t),$$

subject to the initial conditions $x(0) = 1$ and $y(0) = 1$.

We'll solve this one twice, first using elimination (i.e., clever algebra), and then using matrices.

Rearrange the second equation as $x = y' - y$, and then differentiate both sides to get $x' = y'' - y'$. Replace the x and x' in the first equation with these expressions in terms of y to get $y'' - y' = y' - y - 4y$, or $y'' - 2y' + 5y = 0$. So to find y, we have to find the roots of the polynomial $r^2 - 2r + 5 = 0$, which are

$$r = \frac{2 \pm \sqrt{4 - 20}}{2} = 1 \pm 2i$$

and so $y = c_1 e^t \cos 2t + c_2 e^2 \sin 2t$. Next, from $x = y' - y$, we get

$$\begin{aligned} x &= \Big(c_1 e^t \cos 2t - 2c_1 e^t \sin 2t + c_2 e^t \sin 2t + 2c_2 e^t \cos 2t\Big) - \Big(c_1 e^t \cos 2t + c_2 e^t \sin 2t\Big) \\ &= 2c_2 e^t \cos 2t - 2c_1 e^t \sin 2t \end{aligned}$$

From $x(0) = 1$ we conclude $2c_2 = 1$ or $c_2 = \frac{1}{2}$, and from $y(0) = 1$ we conclude $c_1 = 1$. So the solution of the problem is

$$x = e^t \cos 2t - 2e^t \sin 2t, \quad y = e^t \cos 2t + \tfrac{1}{2}e^t \sin 2t.$$

For the matrix version of the solution, we rewrite the system as

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} 1 & -4 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

We need to find the eigenvalues and eigenvectors of the matrix, so we calculate

$$\det \begin{bmatrix} 1-\lambda & -4 \\ 1 & 1-\lambda \end{bmatrix} = (1-\lambda)^2 + 4 = \lambda^2 - 2\lambda + 5$$

which is the same polynomial we had before, so the eigenvalues are $\lambda = 1 \pm 2i$.

To find the eigenvector corresponding to the eigenvalue $\lambda = 1 - 2i$ we need the kernel of the matrix

$$\begin{bmatrix} 2i & -4 \\ 1 & 2i \end{bmatrix}, \quad \text{which is spanned by} \quad \begin{bmatrix} -2i \\ 1 \end{bmatrix}.$$

Two linearly independent solutions of the system are then the real and imaginary parts of

$$e^{(1-2i)t} \begin{bmatrix} -2i \\ 1 \end{bmatrix} = e^t(\cos 2t - i \sin 2t) \begin{bmatrix} -2i \\ 1 \end{bmatrix} = e^t \begin{bmatrix} -2\sin 2t \\ \cos 2t \end{bmatrix} + ie^t \begin{bmatrix} -2\cos 2t \\ -\sin 2t \end{bmatrix}.$$

Thus

$$\begin{bmatrix} x \\ y \end{bmatrix} = e^t \left(c_1 \begin{bmatrix} -2\sin 2t \\ \cos 2t \end{bmatrix} + c_2 \begin{bmatrix} 2\cos 2t \\ \sin 2t \end{bmatrix} \right).$$

Then $x(0) = y(0) = 1$ implies that

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2c_2 \\ c_1 \end{bmatrix}$$

which tells us that $c_2 = \frac{1}{2}$ and $c_1 = 1$, and so the solution of the initial-value problem is

$$\begin{bmatrix} x \\ y \end{bmatrix} = e^t \begin{bmatrix} -2\sin 2t + \cos 2t \\ \cos 2t + \frac{1}{2}\sin 2t \end{bmatrix}.$$

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6. Prove that the solution of the initial-value problem $u'' + cu = 0$, $u(0) = a$, $u'(0) = b$ for $c < 0$ exists (easy — just write it down) and is unique (to do this, "factor" the operator and then apply the theorem on page 4 of the notes twice).

Since $c < 0$, to simplify notation, let $c = -k^2$, so the equation becomes $u'' - k^2 u = 0$ and (since the roots of $r^2 - k^2$ are $r = \pm k$), the general solution of this equation is

$$u = c_1 e^{kt} + c_2 e^{-kt}.$$

Now we need

$$u(0) = c_1 + c_2 = a \qquad \text{and} \qquad u'(0) = kc_1 - kc_2 = b.$$

Some easy algebra yields

$$c_1 = \frac{ka + b}{2k} \qquad \text{and} \qquad c_2 = \frac{ka - b}{2k}$$

and so the solution exists because

$$u(t) = \frac{ka + b}{2k} e^{kt} + \frac{ka - b}{2k} e^{-kt}$$

(with $k = \sqrt{-c}$) works.

To prove uniqueness, we start the usual way and suppose there are two solutions to the problem, call them u_1 and u_2, and let $v(x) = u_1(x) - u_2(x)$. Then v will satisfy:

$$v'' - k^2 v = 0, \quad v(0) = 0, \quad v'(0) = 0.$$

Using the hint, "factor" the differential equation as $(D+k)(D-k)v = 0$, where D stands for taking the derivative. If we let $w = (D-k)v = v' - kv$, then w satisfies

$$(D+k)w = w' + kw = 0, \quad w(0) = v'(0) - kv(0) = 0.$$

This is an initial-value problem for a first order equation, to which the theorem in the notes applies. Therefore the unique solution of this problem is $w = 0$.

Since $w = 0$, we have that v satisfies:

$$v' - kv = 0, \quad v(0) = 0.$$

But this is again an initial-value problem for a first order equation, the unique solution of which is $v = 0$. Therefore the difference between two solutions of the original problem must be zero, so we have proved uniqueness.

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7. (a) *Torricelli's law* states that fluid will leak out of a small hole at the base of a container at a rate proportional to the square root of the height of the fluid's surface from the base. Suppose that a cylindrical container is initially filled to a depth of one foot. If it takes one minute for three quarters of the fluid to leak out, how long will it take for all of the fluid to leak out?

(b) It is desired to design a "water clock" by making a container that is in the shape of some surface of revolution with a small hole in the bottom, so that as the water empties out of the hole, the water level in the container falls at a constant rate. What should be the shape of the container?

(a) First we have to translate the words into a differential equation: "[F]luid leak[s] out . . . at [a] rate" means $\dfrac{dV}{dt}$, where $V(t)$ is the volume of fluid in the tank at time t. "proportional to the square root of the height" means $-k\sqrt{y}$ for some constant k (we put in the minus sign because the volume of fluid in the container is decreasing), and where $y(t)$ is the height of the fluid in the container. So we can write Torricelli's law as

$$\frac{dV}{dt} = -k\sqrt{y}.$$

For a cylindrical tank of radius r, we have $V(t) = \pi r^2 y(t)$, so $\dfrac{dV}{dt} = \pi r^2 \dfrac{dy}{dt}$. So let $K = \dfrac{k}{\pi r^2}$ and we can write the initial-value problem for $y(t)$:

$$\frac{dy}{dt} = -K\sqrt{y}, \quad y(0) = 1, \text{ and we also know that } y(1) = \tfrac{1}{4}$$

(and for the record, y has units of feet, and t has units of minutes).

The differential equation is separable, and we calculate:

$$\frac{dy}{\sqrt{y}} = -K\,dt$$
$$2\sqrt{y} = C - Kt$$

Now $y(0) = 1$ gives us $C = 2$, and then $y(1) = \frac{1}{4}$ says that $1 = 2 - K$, which gives us that $K = 1$. Therefore the solution of the initial-value problem is

$$2\sqrt{y} = 2 - t$$

and it is easy to see that the container will be empty (i.e., $y = 0$) when $t = 2$ minutes.

(b) For a general surface of revolution, say we're rotating the curve $r = f(y)$ around the y-axis, Then the volume up to height y is

$$V(y) = \int_0^y \pi r^2 \, dy = \int_0^y \pi (f(\alpha))^2 \, d\alpha$$

where we have used α as a "dummy variable" in the integral.

Now, the "water level falls at a constant rate" means that $\dfrac{dy}{dt} = C$ for some constant C.

We can combine this with Torricelli's law, $\dfrac{dV}{dt} = k\sqrt{y}$ and the chain rule $\dfrac{dV}{dt} = \dfrac{dV}{dy}\dfrac{dy}{dt}$ to write

$$C\,\frac{dV}{dy} = k\sqrt{y}.$$

Moreover, we know how V depends on y, so we can use the fundamental theorem of calculus to write:

$$\frac{dV}{dy} = \frac{d}{dy}\int_0^y \pi (f(\alpha))^2 \, d\alpha = \pi (f(y))^2.$$

Combining the last two equations gives us

$$C\pi (f(y))^2 = k\sqrt{y}$$

or

$$f(y) = \sqrt{\frac{k\sqrt{y}}{C\pi}} = A\sqrt[4]{y}$$

for a suitably chosen constant A. So the curve that should be rotated to obtain our water clock is given by $r = A\sqrt[4]{y}$, or $y = Br^4$.

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8. For a function $u(x, y)$ of two variables, its Laplacian is defined to be $\Delta u = u_{xx} + u_{yy}$. Which radial functions (i.e., functions of the polar coordinate r but independent of θ) are harmonic (i.e., satisfy the PDE $\Delta u = 0$, or $u_{xx} + u_{yy} = 0$) ?

This is first and foremost an exercise in the chain rule for partial derivatives! If $u(x, y) = f(r)$, where $r^2 = x^2 + y^2$ then

$$\frac{\partial u}{\partial x} = \frac{df}{dr}\frac{\partial r}{\partial x} \quad \text{and} \quad \frac{\partial u}{\partial y} = \frac{df}{dr}\frac{\partial r}{\partial y}.$$

If we take the partial derivative of both sides of $r^2 = x^2 + y^2$ with respect to x. then we get

$$2r\frac{\partial r}{\partial x} = 2x\,, \quad \text{so} \quad \frac{\partial r}{\partial x} = \frac{x}{r}.$$

Similarly,

$$\frac{\partial r}{\partial y} = \frac{y}{r},$$

and so we can write:

$$u_x = f'(r)\frac{x}{r} \quad \text{and} \quad u_y = f'(r)\frac{y}{r}.$$

Now we can take the derivative again, using the product, quotient and chain rules together with the fact that we know $\partial r/\partial x$ and $\partial r/\partial y$:

$$\begin{aligned} u_{xx} &= \frac{\partial}{\partial x}\left(f'(r)\frac{x}{r}\right) \\ &= \left(f''(r)\frac{\partial r}{\partial x}\right)\frac{x}{r} + f'(r)\frac{1}{r} - f'(r)\frac{x}{r^2}\frac{\partial r}{\partial x} \\ &= f''(r)\frac{x^2}{r^2} + f'(r)\frac{1}{r} - f'(r)\frac{x^2}{r^3} \\ &= f''(r)\frac{x^2}{r^2} + \frac{f'(r)}{r}\left(1 - \frac{x^2}{r^2}\right). \end{aligned}$$

Similarly,

$$u_{yy} = f''(r)\frac{y^2}{r^2} + \frac{f'(r)}{r}\left(1 - \frac{y^2}{r^2}\right).$$

Therefore

$$\Delta u = u_{xx} + u_{yy} = f''(r)\left(\frac{x^2+y^2}{r^2}\right) + \frac{f'(r)}{r}\left(2 - \frac{x^2+y^2}{r^2}\right) = f''(r) + \frac{f'(r)}{r}$$

So we have to solve

$$f'' + \frac{1}{r}f' = 0.$$

We do this in two jumps: let $g = f'$, so we need $g' + \frac{1}{r}g = 0$, in other words

$$\frac{dg}{g} = -\frac{dr}{r} \quad \text{which implies} \quad g = \frac{c_1}{r}.$$

Integrate g to get

$$f(r) = c_1 \ln r + c_2.$$

These are the functions we seek (the "radial harmonic functions" on $\mathbb{R}^2$).