1. Suppose $f$ is a function of one variable that has a continuous second derivative. Show that for any constants $a$ and $b$, the function

$$u(x, y) = f(ax + by)$$

is a solution of the PDE

$$u_{xx}u_{yy} - u_{xy}^2 = 0.$$  

This is an exercise in using the chain rule. For instance, $u_x(x, y) = af'(ax + by)$, $u_{xx} = a^2 f''(ax + by)$, etc., so eventually:

$$u_{xx}u_{yy} - u_{xy}^2 = a^2 b^2 (f''(ax + by))^2 - (ab)^2 (f''(ax + by))^2 = 0.$$  

2. Give an example that shows why solutions of the wave equation $u_{tt} = u_{xx}$ do not necessarily satisfy the maximum principle (i.e., give an example of an explicit solution of the equation for which the maximum principle does not hold).

For this, we need a solution to the wave equation for $x \in (0, L)$ and for $t \in (0, T)$ for which the maximum occurs in the interior of the rectangle. For instance, the function

$$u(x, t) = \sin x \sin t$$

satisfies the wave equation, but the maximum of $u = 1$ occurs when $x = t = \pi/2$, in the interior of the rectangle $[0, \pi] \times [0, \pi]$ (where $u = 0$ identically on the boundary of the rectangle).

3. Find the function $u(x, t)$ that satisfies

$$u_t = 2u_{xx}$$

for $(x, t) \in (0, 3) \times (0, \infty)$, together with the initial condition

$$u(x, 0) = \sin \frac{\pi x}{6} + 4 \sin \frac{5\pi x}{6}$$

for $x \in [0, 3]$, and the boundary conditions:

$$u(0, t) = 0 \quad u_x(3, t) = 0$$

for all $t > 0$. (Hint: Look for “separated” solutions.)

A separated solution is of the form $u(x, t) = F(x)G(t)$, and we would need $F(0) = 0$ and $F'(3) = 0$ to satisfy the boundary conditions. For $u$ of this form, the heat equation becomes:

$$F(x)G'(t) = 2F''(x)G(t).$$

Divide both sides by $F(x)G(t)$ and get

$$\frac{G'(t)}{2G(t)} = \frac{F''(x)}{F(x)}.$$
4. Find the closed form (similar to d’Alembert’s formula) of the solution $u(x,t)$ of the initial-boundary value problem for the semi-infinite string:

$$u_{tt} - c^2 u_{xx} = 0 \quad \text{for } x, t > 0$$

where $u(x,0) = f(x)$ for $x > 0$, and $u_t(x,0) = 0$ for $x > 0$, and $u(0,t) = \alpha(t)$ for $t \geq 0$, where $f$ and $\alpha$ are $C^2$ functions and satisfy $f(0) = \alpha(0)$, $\alpha'(0) = 0$ and $\alpha''(0) = c^2 f''(0)$. Verify that the solution is $C^2$ for all $x, t > 0$.

We’ll solve two separate problems here. First, we’ll find $v(x,t)$ that satisfies everything except that $v(0,t) = 0$ instead of $\alpha(t)$. Then we’ll find $w(x,t)$ that satisfies everything except that $v(x,0) = 0$ instead of $f(x)$. Then it’ll be the case that $u(x,t) = v(x,t) + w(x,t)$ is the solution of the whole problem.

First, for $v(x,t)$, start with the d’Alembert form $v(x,t) = F(x+ct) + G(x-ct)$. We need to know values of $F(z)$ for $z > 0$, and of $G(z)$ for both positive and negative values of $z$. We have to reconcile this with $v(x,0) = f(x)$ for $x > 0$, $v_t(x,0) = 0$ and $v(0,t) = 0$ for $t > 0$. These conditions tell us:

$$F(x) + G(x) = f(x) \quad \text{and} \quad F'(x) - G'(x) = 0$$

for $x > 0$, and

$$G(-t) = G(t)$$
for $t > 0$. But from this it’s clear that we should take $F(x) = G(x) = \frac{1}{2} f(x)$ for $x > 0$, and $G(s) = -\frac{1}{2} f(-s)$ if $x < 0$. This gives us:

$$v(x, t) = \begin{cases} \frac{1}{2} f(x + ct) + \frac{1}{2} f(x - ct) & \text{if } x - ct > 0 \\ \frac{1}{2} f(x + ct) - \frac{1}{2} f(ct - x) & \text{if } x - ct < 0 \end{cases}$$

The interpretation of this is that the signal “bounces off” the fixed end of the string at $x = 0$ and is reflected back in “inverted” form.

Next, for $w(x, t)$, start with $v(x, t) = F(x + ct) + G(x - ct)$ as usual, where this time we need $w(x, 0) = 0$ and $w_t(x, 0) = 0$ for $x > 0$ and $w(0, t) = \alpha(t)$ for $t > 0$. These conditions tell us:

$$F(x) + G(x) = 0 \quad \text{and} \quad F'(x) - G'(x) = 0$$

for $x > 0$, so choose $F(x) = G(x) = 0$ for $x > 0$, and

$$G(-ct) = \alpha(t)$$

for $t > 0$, i.e., $G(s) = \alpha(-s/c)$ for $s < 0$. This gives us

$$w(x, t) = \begin{cases} 0 & \text{if } x - ct > 0 \\ \alpha(t - x/c) & \text{if } x - ct < 0 \end{cases}$$

So altogether:

$$u(x, t) = \begin{cases} \frac{1}{2} f(x + ct) + \frac{1}{2} f(x - ct) & \text{if } x - ct > 0 \\ \frac{1}{2} f(x + ct) - \frac{1}{2} f(ct - x) + \alpha(t - x/c) & \text{if } x - ct < 0 \end{cases}$$

That’s it.