Solving the Black-Scholes equation

Now we can divide through by $dt$ to get the Black-Scholes equation:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0.$$ 

In this equation, we’re looking for $V(S,t)$ and the interest rate $r$ and the stock’s volatility $\sigma$ are “known” constants. It’s interesting that the stock’s growth rate $\mu$ doesn’t appear in the equation at all.

Now we (that is, you) need to solve the equation with various “final” conditions at time $T$. In particular, we need to do this for $C$ and $P$ with the conditions given above.

To derive the solution, the main part of the work is to convert the Black-Scholes equation into the usual heat equation. To do this, you’ll have to make three kinds of changes of variable:

- To get the time running in the right direction, you can define a new variable $\tau = T - t$. Then $t = T$ will correspond to $\tau = 0$.
- Since it was $dS/S = d(\log S)$ that satisfied the standard Wiener process that leads to the usual heat equation, it makes sense to define a new variable $x = \log S$ (natural logarithm). This should get rid of the appearances of the independent variable $S$ or $x$ multiplying the various derivatives.
- As we did at the beginning of the course, a substitution of the form $u = e^{\alpha x + \beta \tau} V$ can be used to get rid of unwanted constants and first-order in $x$ terms.

With these hints, show that the value of a European call option with strike price $E$ and expiry time $T$ is given by:

$$C(S,t) = SF(A_+) - Ee^{-r(T-t)}F(A_-)$$

where $F(x)$ is the cumulative distribution function for the standard normal distribution:

$$F(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-p^2/2} dp,$$

and the constants $A_\pm$ are given by

$$A_\pm = \frac{\log(S/E) + (r \pm \frac{1}{2}\sigma^2)(T - t)}{\sigma \sqrt{T - t}}$$

Here we go – first, if $\tau = T - t$ then $\frac{\partial V}{\partial \tau} = -\frac{\partial V}{\partial t}$, so the equation becomes

$$\frac{\partial V}{\partial \tau} = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV$$

which looks more like the (forward) heat equation.
Next, let \( x = \log S \) (or \( S = e^x \)). Then
\[
\frac{\partial V}{\partial S} = \frac{\partial V}{\partial x} \frac{dx}{dS} = \frac{1}{S} \frac{\partial V}{\partial x}
\]
and
\[
\frac{\partial^2 V}{\partial S^2} = \frac{1}{S} \frac{\partial}{\partial x} \left( \frac{1}{S} \frac{\partial V}{\partial x} \right) = \frac{1}{S^2} \frac{\partial^2 V}{\partial x^2} - \frac{1}{S^2} \frac{\partial V}{\partial x}
\]
(because \( 1/S = e^{-x} \) so \( d(1/S)/dx = -e^{-x} = -1/S \)) and the equation becomes
\[
\frac{\partial V}{\partial \tau} = \frac{1}{2} \sigma^2 S^2 \left( \frac{1}{S^2} \frac{\partial^2 V}{\partial x^2} - \frac{1}{S^2} \frac{\partial V}{\partial x} \right) + rS \left( \frac{1}{S} \frac{\partial V}{\partial x} \right) - rV
\]
This last equation is of the form
\[
\frac{\partial V}{\partial \tau} = A \frac{\partial^2 V}{\partial x^2} + B \frac{\partial V}{\partial x} + CV
\]
for certain constants \( A, B \) and \( C \) (with \( A > 0 \)). To avoid dealing with too much algebra, we’ll start by proving the following helpful result:

**Lemma.** Suppose \( V(x, \tau) \) satisfies
\[
\frac{\partial V}{\partial \tau} = A \frac{\partial^2 V}{\partial x^2} + B \frac{\partial V}{\partial x} + CV
\]
with initial condition \( V(x, 0) = f(x) \), where \( A, B \) and \( C \) are constants (with \( A > 0 \)). Then
\[
V(x, \tau) = \frac{e^{C \tau}}{\sqrt{4 \pi A \tau}} \int_{-\infty}^{\infty} e^{-\frac{1}{2} \left( \frac{y-x-\beta \tau}{\sqrt{2 A \tau}} \right)^2} f(y) \, dy.
\]

**Proof:** Let \( u = e^{\alpha x + \beta \tau} V \) Then \( V = e^{-(\alpha x + \beta \tau)} u \), therefore
\[
\frac{\partial V}{\partial \tau} = e^{-(\alpha x + \beta \tau)} \left( \frac{\partial u}{\partial \tau} - \beta u \right)
\]
and
\[
\frac{\partial V}{\partial x} = e^{-(\alpha x + \beta \tau)} \left( \frac{\partial u}{\partial x} - \alpha u \right)
\]
and
\[
\frac{\partial^2 V}{\partial x^2} = e^{-(\alpha x + \beta \tau)} \left( \frac{\partial^2 u}{\partial x^2} - 2\alpha \frac{\partial u}{\partial x} + \alpha^2 u \right)
\]
Putting this into the equation and canceling the \( e^{-(\alpha x + \beta \tau)} \) from both sides gives
\[
\frac{\partial u}{\partial \tau} - \beta u = A \left( \frac{\partial^2 u}{\partial x^2} - 2\alpha \frac{\partial u}{\partial x} + \alpha^2 u \right) + B \left( \frac{\partial u}{\partial x} - \alpha u \right) + Cu
\]
which we can rearrange as
\[
\frac{\partial u}{\partial \tau} = A \frac{\partial^2 u}{\partial x^2} + (B - 2\alpha A) \frac{\partial u}{\partial x} + (C + \beta - \alpha B + \alpha^2 A) u
\]
To get rid of the $\partial u/\partial x$ term, we should choose 
\[ \alpha = \frac{B}{2A}, \]
in which case the equation becomes 
\[ \frac{\partial u}{\partial \tau} = A \frac{\partial^2 u}{\partial x^2} + \left(C + \beta - \frac{B^2}{4A}\right)u \]
And to get rid of the $u$ term, we should choose 
\[ \beta = \frac{B^2}{4A} - C \]
and we arrive at the heat equation 
\[ \frac{\partial u}{\partial \tau} = A \frac{\partial^2 u}{\partial x^2} \]
with $k = A$. And the initial data for $u$ are 
\[ u(x, 0) = e^{\alpha x} V(x, 0) = e^{\frac{B}{2A} x} f(x). \]
And the solution of this initial-value problem is 
\[ u(x, \tau) = e^{\alpha x + \beta \tau} e^{C \tau} \frac{1}{\sqrt{4\pi A \tau}} \int_{-\infty}^{\infty} e^{-\frac{(y-x-B\tau)^2}{4\tau}} e^{\frac{B}{2A} y} f(y) dy \]
Now we need to complete the square in the exponentials, so we calculate 
\[ \frac{(x-y)^2}{4A \tau} - \frac{B}{2A} y = \frac{1}{4A \tau} \left(y^2 - (2x + 2B \tau)y + x^2\right) \]
\[ = \frac{1}{4A \tau} \left(y^2 - 2(x + B \tau)y + (x + B \tau)^2 - 2xB \tau - B^2 \tau^2\right) \]
\[ = \frac{(y-x-B\tau)^2}{4A \tau} - \frac{Bx}{2A} - \frac{B^2 \tau}{4A} \]
\[ = \left(\frac{y-x-B\tau}{2\sqrt{A \tau}}\right)^2 - \alpha x - \beta \tau - C \tau \]
Therefore 
\[ u(x, \tau) = e^{\alpha x + \beta \tau} e^{C \tau} \frac{1}{\sqrt{4\pi A \tau}} \int_{-\infty}^{\infty} e^{-\frac{(y-x-B\tau)^2}{2\sqrt{A \tau}}} e^{\frac{B}{2A} y} f(y) dy \]
Multiply by $e^{-(\alpha x + \beta \tau)}$ and fiddle with the 2’s in the exponential and obtain 
\[ V(x, \tau) = \frac{e^{C \tau}}{\sqrt{4\pi A \tau}} \int_{-\infty}^{\infty} e^{-\frac{1}{2} \left(\frac{y-x-B\tau}{\sqrt{2A \tau}}\right)^2} f(y) dy. \]
This completes the proof of the lemma.

Now we want to apply the lemma with $A = \frac{1}{2} \sigma^2$, with $B = r - \frac{1}{2} \sigma^2$ and with $C = -r$. The result is 
\[ V(x, \tau) = e^{-r \tau} \frac{1}{\sigma \sqrt{2\pi \tau}} \int_{-\infty}^{\infty} e^{-\frac{1}{2} \left(\frac{y-x-(r-\frac{1}{2} \sigma^2) \tau}{\sigma \sqrt{\tau}}\right)^2} f(y) dy. \]
Now we apply the initial data. For a call option, we have

\[ V(S, t = T) = \begin{cases} 
S - E & \text{if } S > E \\
0 & \text{if } S \leq E 
\end{cases} \]

So this is the data for \( \tau = 0 \) and since \( S = e^x \) it makes

\[ f(x) = \begin{cases} 
e^x - E & \text{if } e^x > E \\
0 & \text{if } e^x \leq E 
\end{cases} \]

or in other words the “break point” of \( f(x) \) comes at \( \log E \). So we can rewrite the solution so far as

\[ V(x, \tau) = e^{-r\tau} \int_{\log E}^{\infty} e^{-\frac{1}{2} \left( \frac{y-x-(r-\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}} \right)^2} (e^y - E) \, dy. \]

Now we make the change of variables:

\[ z = y - x - (r - \frac{1}{2}\sigma^2)\tau \quad \text{so} \quad dz = \frac{dy}{\sigma\sqrt{\tau}} \quad \text{and} \quad y = x + \left( r - \frac{1}{2}\sigma^2 \right) \tau + \sigma\sqrt{\tau}z \]

and rewrite the solution as

\[ V(x, \tau) = e^{-r\tau} \int_{\log E - x - (r - \frac{1}{2}\sigma^2)\tau}^{\infty} e^{-\frac{1}{2} z^2} \left( e^{x+(r-\frac{1}{2}\sigma^2)\tau + \sigma\sqrt{\tau}z} - E \right) \, dz. \]

Almost there! There are two terms. We’ll write each one in terms of the cumulative distribution function of the Gaussian, in other words, in terms of the function

\[ F(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-p^2/2} \, dp. \]

It is important to note that

\[ F(\infty) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-p^2/2} \, dp = 1 \]

and \( e^{-p^2/2} \) is even, so by the substitution \( p \to -p \) we can conclude

\[ F(-x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-p^2/2} \, dp, \]

So let’s take the second term of \( V \) first, namely

\[ -E \frac{e^{-r\tau}}{\sqrt{2\pi}} \int_{\log E - x - (r - \frac{1}{2}\sigma^2)\tau}^{\infty} e^{-\frac{1}{2} z^2} \, dx = -Ee^{-r\tau} F \left( \frac{x - \log E + (r - \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}} \right) \]

using the facts that \( \tau = T - t \), that \( x = \log S \) and the definition of \( A_- \) given in the problem.

Now we work on the first term:

\[ e^{-r\tau} \frac{e^{-r\tau}}{\sqrt{2\pi}} \int_{\log E - x - (r - \frac{1}{2}\sigma^2)\tau}^{\infty} e^{-\frac{1}{2} z^2} e^{x+(r-\frac{1}{2}\sigma^2)\tau + \sigma\sqrt{\tau}z} \, dz \]
We can factor $e^z$ out of the integrand as $S$. Then we need to complete the square in the rest of the exponent:

$$-\frac{1}{2}(z^2 - 2\sigma\sqrt{\tau}z - (2r - \sigma^2)\tau) = -\frac{1}{2}((z^2 - \sigma\sqrt{\tau})^2 - \sigma^2\tau - (2r - \sigma^2)\tau)$$

$$= -\frac{1}{2}((z - \sigma\sqrt{\tau})^2 - 2\tau)$$

Using this, we can rewrite the first term as

$$\frac{e^{-r\tau}}{\sqrt{2\pi}} \int_{10g E - x - (r - \frac{1}{2}\sigma^2)\tau}^{\infty} e^{-\frac{1}{2}z^2} e^{x + (r - \frac{1}{2}\sigma^2)\tau + \sigma\sqrt{\tau}z} \, dz = \frac{S}{\sqrt{2\pi}} \int_{10g E - x - (r - \frac{1}{2}\sigma^2)\tau}^{\infty} e^{-\frac{1}{2}(z - \sigma\sqrt{\tau})^2} \, dz$$

Now let $w = z - \sigma\sqrt{\tau}$ so $dw = dz$ and $z = w + \sigma\sqrt{\tau}$ and this becomes

$$\frac{S}{\sqrt{2\pi}} \int_{10g E - x - (r + \frac{1}{2}\sigma^2)\tau}^{\infty} e^{-\frac{1}{2}w^2} \, dw$$

(pay particular attention that $r - \frac{1}{2}\sigma^2$ changed to $r + \frac{1}{2}\sigma^2$ in the lower limit of integration because we added $\sigma\sqrt{\tau}$ in changing from $z$ to $w$). And finally, using that $\tau = T - t$, that $x = \log S$ and the definition of $A_+$ given in the problem, we can rewrite this as

$$\frac{S}{\sqrt{2\pi}} \int_{10g E - x - (r + \frac{1}{2}\sigma^2)\tau}^{\infty} e^{-\frac{1}{2}w^2} \, dw = SF\left(\frac{\log(S/E) + (r + \frac{1}{2}\sigma^2)(T - t)}{\sigma\sqrt{T - t}}\right)$$

$$= SF(A_+)$$

Woohoo!