Solving the Black-Scholes equation

Now we can divide through by dt to get the *Black-Scholes equation*:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0.$$

In this equation, we're looking for $V(S,t)$ and the interest rate r and the stock's volatility σ are "known" constants. It's interesting that the stock's growth rate μ doesn't appear in the equation at all.

Now we (that is, you) need to solve the equation with various "final" conditions at time T. In particular, we need to do this for C and P with the conditions given above.

To derive the solution, the main part of the work is to convert the Black-Scholes equation into the usual heat equation. To do this, you'll have to make three kinds of changes of variable:

- To get the time running in the right direction, you can define a new variable $\tau = T - t$. Then $t = T$ will correspond to $\tau = 0$.
- Since it was $dS/S = d(\log S)$ that satisfied the standard Wiener process that leads to the usual heat equation, it makes sense to define a new variable $x = \log S$ (natural logarithm). This should get rid of the appearances of the independent variable S or x multiplying the various derivatives.
- As we did at the beginning of the course, a substitution of the form $u = e^{\alpha x + \beta\tau} V$ can be used to get rid of unwanted constants and first-order in x terms.

With these hints, show that the value of a European call option with strike price E and expiry time T is given by:

$$C(S,t) = SF(A_+) - Ee^{-r(T-t)}F(A_-)$$

where $F(x)$ is the cumulative distribution function for the standard normal distribution:

$$F(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-p^2/2}\, dp,$$

and the constants $A_\pm$ are given by

$$A_\pm = \frac{\log(S/E) + (r \pm \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

Here we go – first, if $\tau = T - t$ then $\dfrac{\partial V}{\partial \tau} = -\dfrac{\partial V}{\partial t}$, so the equation becomes

$$\frac{\partial V}{\partial \tau} = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV$$

which looks more like the (forward) heat equation.

Next, let $x = \log S$ (or $S = e^x$). Then

$$\frac{\partial V}{\partial S} = \frac{\partial V}{\partial x}\frac{dx}{dS} = \frac{1}{S}\frac{\partial V}{\partial x}$$

and

$$\frac{\partial^2 V}{\partial S^2} = \frac{1}{S}\frac{\partial}{\partial x}\left(\frac{1}{S}\frac{\partial V}{\partial x}\right) = \frac{1}{S^2}\frac{\partial^2 V}{\partial x^2} - \frac{1}{S^2}\frac{\partial V}{\partial x}$$

(because $1/S = e^{-x}$ so $d(1/S)/dx = -e^{-x} = -1/S$) and the equation becomes

$$\begin{aligned}\frac{\partial V}{\partial \tau} &= \frac{1}{2}\sigma^2 S^2\left(\frac{1}{S^2}\frac{\partial^2 V}{\partial x^2} - \frac{1}{S^2}\frac{\partial V}{\partial x}\right) + rS\left(\frac{1}{S}\frac{\partial V}{\partial x}\right) - rV \\ &= \frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial x^2} + \left(r - \frac{1}{2}\sigma^2\right)\frac{\partial V}{\partial x} - rV\end{aligned}$$

This last equation is of the form

$$\frac{\partial V}{\partial \tau} = A\frac{\partial^2 V}{\partial x^2} + B\frac{\partial V}{\partial x} + CV$$

for certain constants A, B and C (with $A > 0$). To avoid dealing with too much algebra, we'll start by proving the following helpful result:

Lemma. *Suppose $V(x,\tau)$ satisfies*

$$\frac{\partial V}{\partial \tau} = A\frac{\partial^2 V}{\partial x^2} + B\frac{\partial V}{\partial x} + CV$$

with initial condition $V(x,0) = f(x)$, where A, B and C are constants (with $A > 0$). Then

$$V(x,\tau) = \frac{e^{C\tau}}{\sqrt{4\pi A\tau}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{y-x-B\tau}{\sqrt{2A\tau}}\right)^2} f(y)\,dy.$$

Proof: Let $u = e^{\alpha x + \beta\tau}V$ Then $V = e^{-(\alpha x+\beta\tau)}u$, therefore

$$\frac{\partial V}{\partial \tau} = e^{-(\alpha x+\beta\tau)}\left(\frac{\partial u}{\partial \tau} - \beta u\right)$$

and

$$\frac{\partial V}{\partial x} = e^{-(\alpha x+\beta\tau)}\left(\frac{\partial u}{\partial x} - \alpha u\right)$$

and

$$\frac{\partial^2 V}{\partial x^2} = e^{-(\alpha x+\beta\tau)}\left(\frac{\partial^2 u}{\partial x^2} - 2\alpha\frac{\partial u}{\partial x} + \alpha^2 u\right)$$

Putting this into the equation and canceling the $e^{-(\alpha x+\beta\tau)}$ from both sides gives

$$\frac{\partial u}{\partial \tau} - \beta u = A\left(\frac{\partial^2 u}{\partial x^2} - 2\alpha\frac{\partial u}{\partial x} + \alpha^2 u\right) + B\left(\frac{\partial u}{\partial x} - \alpha u\right) + Cu$$

which we can rearrange as

$$\frac{\partial u}{\partial \tau} = A\frac{\partial^2 u}{\partial x^2} + (B - 2\alpha A)\frac{\partial u}{\partial x} + \left(C + \beta - \alpha B + \alpha^2 A\right)u$$

To get rid of the $\partial u/\partial x$ term, we should choose

$$\alpha = \frac{B}{2A},$$

in which case the equation becomes

$$\frac{\partial u}{\partial \tau} = A\frac{\partial^2 u}{\partial x^2} + \left(C + \beta - \frac{B^2}{4A}\right) u$$

And to get rid of the u term, we should choose

$$\beta = \frac{B^2}{4A} - C$$

and we arrive at the heat equation

$$\frac{\partial u}{\partial \tau} = A\frac{\partial^2 u}{\partial x^2}$$

with $k = A$. And the initial data for u are

$$u(x,0) = e^{\alpha x} V(x,0) = e^{\frac{B}{2A}x} f(x).$$

And the solution of this initial-value problem is

$$u(x,\tau) = \frac{1}{\sqrt{4\pi A\tau}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4A\tau}} e^{\frac{B}{2A}y} f(y)\, dy$$

Now we need to complete the square in the exponentials, so we calculate

$$\begin{aligned}
\frac{(x-y)^2}{4A\tau} - \frac{B}{2A}y &= \frac{1}{4A\tau}\left(y^2 - (2x + 2B\tau)y + x^2\right) \\
&= \frac{1}{4A\tau}\left(y^2 - 2(x + B\tau)y + (x + B\tau)^2 - 2xB\tau - B^2\tau^2\right) \\
&= \frac{(y - x - B\tau)^2}{4A\tau} - \frac{Bx}{2A} - \frac{B^2\tau}{4A} \\
&= \left(\frac{y - x - B\tau}{2\sqrt{A\tau}}\right)^2 - \alpha x - \beta\tau - C\tau
\end{aligned}$$

Therefore

$$u(x,\tau) = \frac{e^{\alpha x + \beta\tau} e^{C\tau}}{\sqrt{4\pi A\tau}} \int_{-\infty}^{\infty} e^{-\left(\frac{y-x-B\tau}{2\sqrt{A\tau}}\right)^2} f(y)\, dy$$

Multiply by $e^{-(\alpha x + \beta\tau)}$ and fiddle with the 2's in the exponential and obtain

$$V(x,\tau) = \frac{e^{C\tau}}{\sqrt{4\pi A\tau}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{y-x-B\tau}{\sqrt{2A\tau}}\right)^2} f(y)\, dy.$$

This completes the proof of the lemma.

Now we want to apply the lemma with $A = \frac{1}{2}\sigma^2$, with $B = r - \frac{1}{2}\sigma^2$ and with $C = -r$. The result is

$$V(x,\tau) = \frac{e^{-r\tau}}{\sigma\sqrt{2\pi\tau}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{y-x-(r-\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}\right)^2} f(y)\, dy.$$

Now we apply the initial data. For a call option, we have

$$V(S,t=T)=\begin{cases} S-E & \text{if } S>E \\ 0 & \text{if } S\le E \end{cases}$$

So this is the data for $\tau=0$ and since $S=e^x$ it makes

$$f(x)=\begin{cases} e^x-E & \text{if } e^x>E \\ 0 & \text{if } e^x\le E \end{cases}$$

or in other words the "break point" of $f(x)$ comes at $\log E$. So we can rewrite the solution so far as

$$V(x,\tau)=\frac{e^{-r\tau}}{\sigma\sqrt{2\pi\tau}}\int_{\log E}^{\infty} e^{-\frac{1}{2}\left(\frac{y-x-(r-\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}\right)^2}\left(e^y-E\right)\,dy.$$

Now we make the change of variables:

$$z=\frac{y-x-(r-\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}} \quad \text{so} \quad dz-\frac{dy}{\sigma\sqrt{\tau}} \quad \text{and} \quad y=x+\left(r-\frac{1}{2}\sigma^2\right)\tau+\sigma\sqrt{\tau}z$$

and rewrite the solution as

$$V(x,\tau)=\frac{e^{-r\tau}}{\sqrt{2\pi}}\int_{\frac{\log E-x-(r-\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}}^{\infty} e^{-\frac{1}{2}z^2}\left(e^{x+(r-\frac{1}{2}\sigma^2)\tau+\sigma\sqrt{\tau}z}-E\right)dz.$$

Almost there! There are two terms. We'll write each one in terms of the cumulative distribution function of the Gaussian, in other words, in terms of the function

$$F(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-p^2/2}\,dp.$$

It is important to note that

$$F(\infty)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-p^2/2}\,dp=1$$

and $e^{-p^2/2}$ is even, so by the substitution $p\to -p$ we can conclude

$$F(-x)=\frac{1}{\sqrt{2\pi}}\int_{x}^{\infty} e^{-p^2/2}\,dp,$$

So let's take the second term of V first, namely

$$\begin{aligned}
-E\frac{e^{-r\tau}}{\sqrt{2\pi}}\int_{\frac{\log E-x-(r-\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}}^{\infty} e^{-\frac{1}{2}z^2}\,dx &= -Ee^{-r\tau}F\left(\frac{x-\log E+(r-\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}\right) \\
&= -Ee^{-r(T-t)}F\left(\frac{\log(S/E)+(r-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right) \\
&= -Ee^{-r(T-t)}F(A_-)
\end{aligned}$$

using the facts that $\tau=T-t$, that $x=\log S$ and the definition of A_- given in the problem.

Now we work on the first term:

$$\frac{e^{-r\tau}}{\sqrt{2\pi}}\int_{\frac{\log E-x-(r-\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}}^{\infty} e^{-\frac{1}{2}z^2}e^{x+(r-\frac{1}{2}\sigma^2)\tau+\sigma\sqrt{\tau}z}\,dz$$

We can factor e^x out of the integrand as S. Then we need to complete the square in the rest of the exponent:

$$-\frac{1}{2}\Big(z^2 - 2\sigma\sqrt{\tau}z - (2r-\sigma^2)\tau\Big) = -\frac{1}{2}\Big((z^2 - \sigma\sqrt{\tau})^2 - \sigma^2\tau - (2r-\sigma^2)\tau\Big)$$
$$= -\frac{1}{2}\Big((z-\sigma\sqrt{\tau})^2 - 2r\tau\Big)$$

Using this, we can rewrite the first term as

$$\frac{e^{-r\tau}}{\sqrt{2\pi}}\int_{\frac{\log E - x - (r-\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}}^{\infty} e^{-\frac{1}{2}z^2} e^{x+(r-\frac{1}{2}\sigma^2)\tau+\sigma\sqrt{\tau}z}\,dz = \frac{S}{\sqrt{2\pi}}\int_{\frac{\log E - x - (r-\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}}^{\infty} e^{-\frac{1}{2}(z-\sigma\sqrt{\tau})^2}\,dz$$

Now let $w = z - \sigma\sqrt{\tau}$ so $dw = dz$ and $z = w + \sigma\sqrt{\tau}$ and this becomes

$$\frac{S}{\sqrt{2\pi}}\int_{\frac{\log E - x - (r+\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}}^{\infty} e^{-\frac{1}{2}w^2}\,dw$$

(pay particular attention that $r - \frac{1}{2}\sigma^2$ changed to $r + \frac{1}{2}\sigma^2$ in the lower limit of integration because we added $\sigma\sqrt{\tau}$ in changing from z to w). And finally, using that $\tau = T - t$, that $x = \log S$ and the definition of A_+ given in the problem, we can rewrite this as

$$\frac{S}{\sqrt{2\pi}}\int_{\frac{\log E - x - (r+\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}}^{\infty} e^{-\frac{1}{2}w^2}\,dw = SF\left(\frac{\log(S/E) + (r+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right)$$
$$= SF(A_+)$$

Woohoo!