d'Alembert's solution of the wave equation / energy

We've derived the one-dimensional wave equation

$$u_{tt} = \frac{T}{\rho}\, u_{xx} = c^2 u_{xx}$$

and now it's time to solve it.

For our first pass, we'll assume that the string is "infinite" and solve the initial-value problem for the equation for $-\infty < x < \infty$ and $t > 0$, together with initial data

$$u(x, 0) = \varphi(x) \qquad u_t(x, 0) = \psi(x)$$

specifying the initial displacement and velocity of the string.

In class we solved this by making a clever change of variables ($p = x + ct$, $q = x - ct$) but now we'll think about finding "traveling wave" solutions with as-yet-to-be-determined velocity k:

$$u(x, t) = f(x - kt)$$

and determine the possible value of k from the wave equation:

If $u(x, t) = f(x - kt)$ then $u_{xx} = f''(x - kt)$ and $u_{tt} = k^2 f''(x - kt)$. Therefore we need

$$0 = u_t t - c^2 u_{xx} = k^2 f''(x - kt) - c^2 f''(x - kt) = (k^2 - c^2) f''(x - kt)$$

Since this must hold for all x and t, we either need $f'' \equiv 0$, i.e., f is a linear function (not very interesting for the *wave* equation) or else we need $k^2 - c^2 = 0$, i.e., $k = \pm c$. So we arrive at the solution we've seen in class (since we can add two solutions to get another one):

$$u(x, t) = f(x - ct) + g(x + ct).$$

Initial-value problem

Since the wave equation is second-order in time, it tells us about acceleration. Therefore it makes sense that we need to specify the initial position and velocity of the medium (string) to get a unique solution: In other words, we seek a (the) solution of the initial-value problem:

$$u_{tt} = c^2 u_{xx} \quad \text{together with} \quad u(x, 0) = \varphi(x) \quad u_t(x, 0) = \psi(x).$$

We start by writing

$$u(x, t) = f(x - ct) + g(x + ct), \tag{1}$$

which "uses up" the PDE, and now we have to choose f and g so that the initial conditions are satisfied. Substituting $f(x - ct) + g(x + ct)$ for u in the two initial conditions gives us

$$f(x) + g(x) = \varphi(x) \tag{2}$$

and

$$-cf'(x) + cg'(x) = \psi(x) \tag{3}$$

Take the derivative of equation (2) to obtain

$$f'(x) + g'(x) = \varphi'(x) \tag{4}$$

and we can view (3) and (4) together as an algebraic system of two equations in the two unknowns $f'(x)$ and $g'(x)$, since $\varphi(x)$, $\psi(x)$ and c are known (or given) quantities in the PDE and the initial data. To solve it, add c times equation (4) to equation (3) to cancel the $f'(x)$ terms and get that

$$2cg'(x) = c\varphi'(x) + \psi(x). \tag{5}$$

Then subtract equation (3) from c times equation (4) to cancel the $g'(x)$ terms and get that

$$2cf'(x) = c\varphi'(x) - \psi(x). \tag{6}$$

After dividing by $2c$ and integrating, equation (5) tells us that

$$g(x) = \frac{1}{2}\varphi(x) + \frac{1}{2c}\int_*^x \psi(s)\,ds$$

where the asterisk means we start the integral "somewhere", which accounts for the constant of integration (and s is the "dummy" variable of integration). Likewise, equation (6) tells us that

$$f(x) = \frac{1}{2}\varphi(x) - \frac{1}{2c}\int_*^x \psi(s)\,ds.$$

Therefore

$$u(x,t) = f(x-ct) + g(x+ct) = \frac{1}{2}\Big(\varphi(x-ct) + \varphi(x+ct)\Big) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds \tag{7}$$

where having $x-ct$ as the lower limit of the integral accounts for the minus sign in the $f(x)$ equation, and while there might have been an additional constant of integration, we will check that $u(x,t)$ defined by this formula satisfies the initial-value problem.

First off, $u(x,t)$ in equation (7) is clearly a function of $x+ct$ plus a function of $x-ct$, so it automatically satisfies the wave equation. So we only need to check the initial conditions:

$$u(x,0) = \frac{1}{2}\Big(\varphi(x) + \varphi(x)\Big) + \frac{1}{2c}\int_x^x \psi(s)\,ds = \varphi(x)$$

since the integral over an interval of length zero vanishes. Also,

$$u_t(x,0) = \frac{1}{2}\Big(-c\varphi'(x) + c\varphi'(x)\Big) + \frac{1}{2c}\Big(c\psi(x) + c\psi(x)\Big) = \psi(x)$$

by the fundamental theorem of calculus and the chain rule.

So we have shown that equation (7) gives a solution of the initial-value problem for the wave equation. It is usually referred to as d'Alembert's solution, since he first wrote about it in the 1740s.

It is useful to take a moment to inspect the solution a bit in order to draw some conclusions about "causality" and the speed of signal propagation (I looked up the spelling!). First, consider a point x_0 on the string at time t_0 — we can ask what part of the initial data affect the value of the solution at the space-time point (x_0, t_0). The part of the solution that involves the initial position is just $\frac{1}{2}\Big(\varphi(x_0 - ct_0) + \varphi(x_0 + ct_0)\Big)$, so the values of φ at only the two points $x = x_0 - ct_0$ and $x = x_0 + ct_0$ affect the value of the solution at (x_0, t_0). The part of the solution that involves the initial velocity is

$$\frac{1}{2c}\int_{x_0-ct_0}^{x_0+ct_0} \psi(s)\, ds$$

so the only values of ψ at points in the x-interval from $x_0 - ct_0$ to $x_0 + ct_0$ affect the value of $u(x_0, t_0)$. Because of this, the *closed* interval $[x_0 - ct_0\,,\, x_0 + ct_0]$ is called the *domain of dependence* for the point (x_0, t_0).

We can flip this reasoning around, and ask, given a point x_0, for which points (x, t) is the value of $u(x, t)$ affected by the initial data at x_0? From the reasoning in the previous paragraph, these will be the points for which $x - ct \le x_0 \le x + ct$, in other words, the points for which

$$x_0 - ct \le x \le x_0 + ct$$

(to conclude this, solve the two preceding inequalities separately for x). The set of points (x, t) with $t > 0$ for which this is true form a triangular wedge in the xt-plane having sides emanating from the point $(x_0, 0)$ with slopes $\pm 1/c$. This region in the xt-plane is called the *region of influence* of the point x_0. The fact that x_0 influences only points within a distance ct from it at time t tells us that *signals propagate with finite speed*, [at most] c for solutions of the wave equation. And this is consistent with our experience of sound and other waves.

Energy conservation and inequalities

Since u_t represents the velocity of the string, it is reasonable to relate

$$\frac{1}{2}\int_{-\infty}^{\infty} u_t^2(x,t)\, dx$$

to the total kinetic energy of the string at time t. It is somewhat less obvious that the quantity

$$\frac{1}{2}\int_{-\infty}^{\infty} c^2 u_x^2(x,t)\, dx$$

should be associated with potential energy, but we can conclude that if it starts out finite, the quantity

$$e(t) = \int_{-\infty}^{\infty} \frac{1}{2}\left(u_t^2(x,t) + c^2 u_x^2(x,t)\right)\, dx$$

remains constant, which implies *conservation of energy.* In turn, this conservation of energy implies that the solution of the wave equation is unique. We'll show this now.

Conservation of energy: *If the quantity*

$$e(0) = \frac{1}{2}\int_{-\infty}^{\infty} \psi(x)^2 + c^2\varphi'(x)^2\,dx$$

is finite, then so will be the quantity

$$e(t) = \frac{1}{2}\int_{-\infty}^{\infty} u_t(x,t)^2 + c^2 u_x(x,t)^2\,dx$$

for all t, and moreover $e(t) = e(0)$ for all t, i.e., $e(t)$ is constant.

Proof. If the initial data $\varphi(x)$ and $\psi(x)$ are bounded functions that are zero outside a finite interval $[a, b]$, then the convergence of the energy integral $e(0)$ for them is not an issue, and then from the range of influence discussion above, we know that at time t, the d'Alembert solution $u(x, t)$ will be bounded and zero outside the x-interval $[a - ct, b + ct]$, so the convergence of the energy integral $e(t)$ is not an issue either. At the end of these notes we'll discuss what happens if we only assume that the energy integral $e(0)$ converges (allowing φ and ψ to be nonzero, although necessarily decaying to zero, on the whole x-axis).

To show that $e(t)$ is a constant, we will show that $\frac{de}{dt} \equiv 0$. To do this, we will use that u satisfies the wave equation and integration by parts as follows:

$$\frac{de}{dt} = \frac{d}{dt}\int_{-\infty}^{\infty} \tfrac{1}{2}u_t^2 + \tfrac{1}{2}c^2u_x^2\,dx = \int_{-\infty}^{\infty} \frac{\partial}{\partial t}(\tfrac{1}{2}u_t^2 + \tfrac{1}{2}c^2u_x^2)\,dx = \int_{-\infty}^{\infty} u_tu_{tt} + c^2u_xu_{xt}\,dx.$$

We're going to integrate the second term of this by parts. Writing the integration by parts formula as

$$\int f\,dg = fg - \int g\,df$$

we take $f = u_x$ and $dg = u_{xt}\,dx = u_{tx}\,dx$ (Keep in mind that the variable of integration is x, and we are treating t as a constant for the purposes of doing the integral). Then $df = u_{xx}\,dx$ and $g = u_t$, so we get that

$$\int_{-\infty}^{\infty} u_xu_{xt}\,dx = u_t(x)u_x(x)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} u_{xx}u_t\,dx$$

If we continue to assume that $u(x, t)$ is zero outside the x-interval $[a - ct\,,\, b + ct]$ (so u_x and u_t will be zero there also), the term being evaluated at $-\infty$ and $+\infty$ (i.e., the "boundary term") will be zero, and so we can insert this into our most recent expression for the derivative of $e(t)$:

$$\frac{de}{dt} = \int_{-\infty}^{\infty} u_tu_{tt} + c^2u_xu_{xt}\,dx. = \int_{-\infty}^{\infty} u_tu_{tt} - c^2u_{xx}u_t\,dx = \int_{-\infty}^{\infty} u_t(u_{tt} - c^2u_{xx})\,dx$$

But the last expression in parenthesis is identically zero because u satisfies the wave equation. Therefore we have shown that

$$\frac{de}{dt} = 0$$

so $e(t)$ is constant (i.e., energy is conserved).

Uniqueness. *There is only one finite-energy solution of the initial-value problem for the wave equation.*

Proof. Suppose there are two different finite-energy solutions of the initial-value problem

$$u_t t = c^2 u_{xx} \qquad u(x,0) = \varphi(x) \qquad u_t(x,0) = \psi(x),$$

call them $u_1(x,t)$ and $u_2(x,t)$. Then their difference, $v(x,t) = u_1(x,t) - u_2(x,t)$ certainly satisfies the (linear!) wave equation, and the initial data for v are

$$v(x,0) = u_1(x,0) - u_2(x,0) = \varphi(x) - \varphi(x) = 0$$

and

$$v_t(x,0) = (u_1)_t(x,0) - (u_2)_t(x,0) = \psi(x) - \psi(x) = 0.$$

Therefore the energy corresponding to $v(x,t)$ is zero at time $t = 0$, and by conservation of energy it is always zero. But then we can conclude that $v_x(x,t) = v_t(x,t) = 0$ for all x and t — hence v is a constant function. And since $v(x,0) = 0$, $v(x,t)$ must be identically zero. Therefore there is no difference between the two solutions (any two finite-energy solutions) of the initial-value problem.

A more refined look at energy. We can remove the finite-energy hypothesis in our uniqueness statement by looking a little more carefully at the idea of the domain of dependence and using Green's theorem in a clever way. From the discussion of domain of dependence (the solution at (x_0, t_0) depends only on the initial data in the interval $[x_0 - ct_0\,,\, x_0 + ct_0]$), it is not hard to conclude that the solution on the x-interval $a \le x \le b$ at time t_0 depends only on the initial data in the interval $[a - ct_0\,,\, b + ct_0]$. It would follow then that the energy at time $t = t_0$ in $[a, b]$ should be no greater than the energy at time $t = 0$ in the interval $[a - ct_0\,,\, b + ct_0]$ (it might be less, since some of the initial energy would be elsewhere on the x axis).

To translate all this into formulas, we seek to show that

$$\int_a^b \tfrac{1}{2}(u_t(s,t_0)^2 + c^2 u_x(s,t_0)^2)\,ds \le \int_{a-ct_0}^{b+t_0} \tfrac{1}{2}(u_t(s,0)^2 + c^2 u_x(s,0)^2)\,ds$$

To do this, we're going to use Green's theorem in a very clever way. Recall that Green's theorem is the following (it might look a tiny bit different from what you're used to, since I've changed the variable usually called y to t):

$$\iint\limits_R \frac{\partial N}{\partial x} - \frac{\partial M}{\partial t}\,dx\,dt = \oint_{\mathrm{bd}(R)} M\,dx + N\,dt$$

where M and N are functions of x and t, R is a region in the xt-plane, and $\mathrm{bd}(R)$ is its boundary, traversed counterclockwise.

We're going to apply Green's theorem to the trapezoid in the xt-plane with vertices $(a - ct_0, 0)$, $(b + ct_0, 0)$, (b, t_0) and (a, t_0) and to the functions $M = \frac{1}{2}(u_t^2 + c^2 u_x^2)$ and $N = c^2 u_x u_t$. It's easy to see where M comes from — that was our energy integrand from before. N is chosen so that the double integral in Green's theorem will be zero, because

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial t} = c^2(u_{xx}u_t + u_x u_{tx}) - (u_t u_{tt} + c^2 u_x u_{xt}) = u_t(c^2 u_{xx} - u_{tt}) = 0$$

because $u_{xt} = u_{tx}$ and because u satisfies the wave equation. So the double integral on the left side of Green's theorem is zero — now we have to evaluate the line integral on the right side. It has four parts:

- We'll start at the bottom left corner of the trapezoid, starting at $(a-ct_0, 0)$. The first segment goes from there to $(b+ct_{,0}\,0)$. Parametrize it as $x = s$, $y = 0$ for $a - ct_0 \le s \le b + ct_0$. Then $dx = ds$ and $dy = 0$, so this part of the integral becomes

$$\int_{a-ct_0}^{b+ct_0} \tfrac{1}{2}\Big((u_t(s,0))^2 + c^2(u_x(s,0))^2\Big)\,ds$$

  in other words, it is the energy at time $t = 0$ in the interval $[a - ct_0\,,\, b + ct_0]$.

- Next,we have to integrate the segment starting at $(b + ct_0, 0)$ and ending at (b, t_0). We can parametrize this as $x = b + c(t_0 - s)$, $y = s$ for $0 \le s \le t_0$. Then $dx = -c\,ds$ and $dy = ds$ and this part of the integral becomes

$$\int_0^{t_0} \frac{-c}{2}\Big(u_t(b+c(t_0-s),s))^2 + c^2(u_x(b+c(t_0-s),s))^2\Big) + c^2 u_x(b+c(t_0-s),s)u_t(b+c(t_0-s),s)\,ds$$

  To process this, let's rewrite it without the indication of where to evaluate u_t and u_x each time:

$$\begin{aligned}\int_0^{t_0} \frac{-c}{2}(u_t^2 + u_x^2) + c^2 u_t u_x\,ds &= -c\int_0^{t_0} \tfrac{1}{2}u_t^2 - c u_t u_x + \tfrac{1}{2}c^2 u_x^2\,ds \\ &= -\frac{c}{2}\int_0^{t_0} (u_t - c u_x)^2\,ds\end{aligned}$$

  Therefore this part of the line integral is ≤ 0.

- The third part of the integral goes along the segment from (b, t_0) to (a, t_0). It can be parametrized as $x = s$, $y = t_0$ for s going from b to a (we know that $a < b$ but that's okay), so we have $dx = ds$ and $dy = 0$. This part of the integral becomes:

$$\int_b^a \tfrac{1}{2}(u_t(s,t_0)^2 + c^2 u_x(s,t_0)^2)\,ds = -\int_a^b \tfrac{1}{2}(u_t(s,t_0)^2 + c^2 u_x(s,t_0)^2)\,ds$$

  in other words, it is the negative of the energy in the interval $[a, b]$ at time $t = t_0$.

- Finally, we have to integrate the segment rom (a, t_0) back to the point $(a - ct_0, 0)$. It can be parametrized as $x = a - c(t_0 - s)$, $y = s$ for s going from t_0 to 0 (again s is decreasing but that's okay). We have $dx = c\,ds$ and $dy = ds$ so this part of the line integral becomes

$$\int_{t_0}^{0} \frac{c}{2}\Big(u_t(a-c(t_0-s),s))^2 + c^2(u_x(a-c(t_0-s),s))^2\Big) + c^2 u_x(a-c(t_0-s),s)u_t(a-c(t_0-s),s)\,ds$$

And once again, to process this, we rewrite it without the indication of where to evaluate u_t and u_x each time:

$$\int_{t_0}^{0} \frac{c}{2}(u_t^2 + u_x^2) + c^2 u_t u_x \, ds = c \int_{t_0}^{0} \tfrac{1}{2}u_t^2 + c u_t u_x + \tfrac{1}{2}c^2 u_x^2 \, ds$$

$$= \frac{c}{2} \int_{t_0}^{0} (u_t + c u_x)^2 \, ds$$

$$= -\frac{c}{2} \int_{0}^{t_0} (u_t + c u_x)^2 \, ds$$

So this part of the integral is non-positive (just like the second part above).

Because the line integral evaluates to zero, we can summarize our calculations as follows:

$$\boxed{\text{Energy in } [a - ct_0, b + ct_0] \text{ at } t = 0} - \boxed{\text{Energy in } [a, b] \text{ at } t = t_0} = \text{something non-negative}$$

This is the essential energy inequality — it allows us to conclude that if the initial data are zero, there cannot be *any* solution of the initial value problem other than $u \equiv 0$ (finite energy or not) because we can conclude that the energy remains zero in any trapezoid with its base on the x-axis and its left and right slanted sides of slopes $1/c$ and $-1/c$ respectively. And we can cover the xt-plane with such trapezoids, so the only possible solution of the initial value problem is the zero solution. So we have shown:

Uniqueness. *There is only one solution of the initial-value problem for the wave equation.*

Postscript: What if the initial data are not zero outside a finite interval? This follows from the refined version of the energy inequality. As long as the total energy

$$e(0) = \frac{1}{2} \int_{-\infty}^{\infty} \psi(x)^2 + c^2 \varphi'(x)^2 \, dx$$

is finite, then the number $e(0)$ serves as an upper bound for the energy on any finite interval $[a, b]$ at time $t = t_0$

$$\frac{1}{2} \int_{a}^{b} u_t(x, t_0)^2 + c^2 u_x(x, t_0)^2 \, dx$$

because we now know that

$$\frac{1}{2} \int_{a}^{b} u_t(x, t_0)^2 + c^2 u_x(x, t_0)^2 \, dx \leq \frac{1}{2} \int_{a - ct_0}^{b + ct_0} u_t(x, 0)^2 + c^2 u_x(x, 0)^2 \, dx \leq \frac{1}{2} \int_{-\infty}^{\infty} \psi(x)^2 + c^2 \varphi'(x)^2 \, dx = e(0).$$

But now we can let a and b go to $-\infty$ and $+\infty$ respectively and we know that the value of the integral will increase, but be bounded above. Therefore it must converge to a finite value and so the quantity

$$e(t_0) = \frac{1}{2} \int_{-\infty}^{\infty} u_t(x, t)^2 + c^2 u_x(x, t)^2 \, dx$$

will exist and be finite. And then the reasoning from the finite-energy case applies.