Math 425 / AMCS 525 **Dr. DeTurck**
Midterm Exam 1 **February 25, 2016**

There are five problems on this test. The last one is long, but that's because it leads you through the proof of a fairly significant result.

You may use your book and your notes during this exam. Do as much of it as you can during the class period, and turn your work in at the end. But take the sheet with the problems home with you, and you may (re)work any problems you like and turn them in on Tuesday for additional credit.

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1. Solve the following initial-value problem for $u(x, y)$:

$$yu_x + u_y = x \qquad u(x, 0) = x^2$$

In what domain is your solution determined by the initial data?

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2. Let $\varphi(x) = \begin{cases} 1 & \text{if } 0 < x < L \\ 0 & \text{if } x > L \end{cases}$

(a) Solve the heat equation $u_t = ku_{xx}$ for $x > 0$, $t > 0$ with initial condition $u(x, 0) = \varphi(x)$ for $x > 0$ and boundary condition $u_x(0, t) = 0$ for $t > 0$. Express your solution in terms of the error function, Erf.

(b) Solve the wave equation $u_{tt} = c^2 u_{xx}$ for $x > 0$ and $t > 0$, with initial conditions $u(x, 0) = \varphi(x)$ and $u_t(x, 0) = 0$ for $x > 0$ and boundary condition $u_x(0, t) = 0$ for $t > 0$.

(c) Solve the wave equation $u_{tt} = c^2 u_{xx}$ for $x > 0$ and $t > 0$ with initial conditions $u(x, 0) = 0$ and $u_t(x, 0) = \varphi(x)$ for $x > 0$ and boundary conditions $u_x(0, t) = 0$ for $t > 0$.

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3. Find all functions $\varphi(x)$ for which the "traveling wave" $u(x, t) = \varphi(x - ct)$ is a solution of the *heat* equation $u_t = ku_{xx}$. Repeat for the "heat wave" traveling in the other direction: $u(x, t) = \varphi(x + ct)$.

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4. Solve the modified (damped) wave equation:

$$u_{tt} + 2u_t + u = u_{xx}$$

on the whole line with initial data $u(x, 0) = xe^{-x^2}$ and $u_t(x, 0) = 1$. (*Hint*: Consider $w(x, t) = e^t u(x, t)$)

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5. The purpose of this problem is to show that a solution of the heat equation on a finite interval with constant boundary conditions will converge as $t \to \infty$ to the equilibrium solution. We need a preliminary result, though, and the first few parts of this problem lead up to that (it might seem irrelevant at first, but bear with it). In any part of the problem, you may assume that the results of the previous parts are true, even if you skip them. In particular, you might want to start with part (f), and use the result of part (e) when needed.

(a) Let $p(x) = ax^2 + bx + c$ be a quadratic polynomial with the property that $p(x) \geq 0$ for all x (so p does not have two distinct real roots). Explain why this implies that $b^2 \leq 4ac$.

(b) Now consider two (differentiable) functions $u(x)$ and $v(x)$ defined on the interval $0 \leq x \leq L$. Then define

$$p(\lambda) = \int_0^L (|u(x)| + \lambda|v(x)|)^2 \, dx.$$

Once you multiply this out, you can see that $p(\lambda)$ is a quadratic polynomial. Use this together with part (a) to show that

$$\left(\int_0^L |u(x)v(x)| \, dx \right)^2 \leq \int_0^L u(x)^2 \, dx \int_0^L v(x)^2 \, dx.$$

(c) Next, suppose that $f(x)$ is a differentiable function on the interval $0 \leq x \leq L$, and suppose further that $f(0) = f(L) = 0$ (Dirichlet!). Explain why

$$|f(x)| \leq \int_0^L |f'(x)| \, dx$$

for all x between 0 and L.

(d) The quantity on the right side of part (c) is just a number (i.e., a constant). So square both sides, and integrate them from 0 to L and explain why

$$\int_0^L (f(x))^2 \, dx \leq L \left(\int_0^L |f'(x)| \, dx \right)^2$$

(e) Now use the result of part (b) (with $u(x) = f'(x)$ and $v(x) = 1$) to explain why

$$\int_0^L (f(x))^2 \, dx \leq L^2 \int_0^L (f'(x))^2 \, dx.$$

Here comes the PDE part: Let $u(x,t)$ be the solution of the heat equation

$$u_t = ku_{xx} \qquad \text{for } t > 0 \text{ and } 0 < x < L,$$

with

$$\text{initial condition } u(x,0) = \varphi(x) \text{ for } 0 < x < L$$

and

$$\text{(Dirichlet) boundary conditions } u(0,t) = u(L,t) = 0 \text{ for } t > 0.$$

As usual, define the energy function $E(t)$ via

$$E(t) = \frac{1}{2}\int_0^L (u(x,t))^2\,dx.$$

so that $E(t) \geq 0$ for all $t \geq 0$ and $E(0) = \displaystyle\int_0^L (\varphi(x))^2\,dx$.

(f) Use the differential equation, integration by parts, and the result of part (e) (with $f(x) = u(x,t)$ for a fixed value of t) to show that

$$\frac{dE}{dt} \leq -\frac{2k}{L^2}E$$

(g) Now divide both sides by E (which we know is positive), integrate from 0 to t and exponentiate both sides to conclude that

$$0 \leq E(t) \leq E(0)e^{-2kt/L^2}.$$

(h) Conclude that $E(t) \to 0$ as $t \to \infty$, and hence $u(x,t) \to 0$ as $t \to \infty$.

(i) Now consider the initial/boundary value problem

$$u_t = ku_{xx} \qquad \text{for } t > 0 \text{ and } 0 < x < L,$$

with

$$\text{initial condition } u(x,0) = \varphi(x) \text{ for } 0 < x < L$$

and

$$\text{(Dirichlet) boundary conditions } u(0,t) = p \text{ and } u(L,t) = q \text{ for } t > 0.$$

Find the equilibrium solution (the time-independent solution of the PDE that satisfies the boundary conditions, ignoring the initial conditions), and then prove that no matter what the initial condition $\varphi(x)$ is, the solution $u(x,t)$ will approach the equilibrium solution as $t \to \infty$.