Math 425 / AMCS 525 **Dr. DeTurck**
Midterm Exam 1 **February 25, 2016**

There are five problems on this test. The last one is long, but that's because it leads you through the proof of a fairly significant result.

You may use your book and your notes during this exam. Do as much of it as you can during the class period, and turn your work in at the end. But take the sheet with the problems home with you, and you may (re)work any problems you like and turn them in on Tuesday for additional credit.

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1. Solve the following initial-value problem for $u(x,y)$:

$$yu_x + u_y = x \qquad u(x,0) = x^2$$

In what domain is your solution determined by the initial data?

Use the method of characteristics and solve the system:

$$\frac{dx}{ds} = y \qquad \frac{dy}{ds} = 1 \qquad \frac{du}{ds} = x$$

with initial conditions

$$x(0) = t \qquad y(0) = 0 \qquad u(0) = t^2.$$

The middle equation tells us that $y = s$, and then the first equation becomes $x' = s$ so $x = \frac{1}{2}s^2 + t$. Finally, we have $u' = \frac{1}{2}s^2 + t$ so $u = \frac{1}{6}s^3 + st + t^2$. To unwind stuff, we have $s = y$ and then $t = x - \frac{1}{2}y^2$, so

$$u(x,y) = \frac{y^3}{6} + xy - \frac{y^3}{2} + \left(x - \frac{y^2}{2}\right)^2 = xy - \frac{y^3}{3} + x^2 - xy^2 + \frac{y^4}{4}.$$

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2. Let $\varphi(x) = \begin{cases} 1 & \text{if } 0 < x < L \\ 0 & \text{if } x > L \end{cases}$

(a) Solve the heat equation $u_t = ku_{xx}$ for $x > 0$, $t > 0$ with initial condition $u(x,0) = \varphi(x)$ for $x > 0$ and boundary condition $u_x(0,t) = 0$ for $t > 0$. Express your solution in terms of the error function, Erf.

(b) Solve the wave equation $u_{tt} = c^2 u_{xx}$ for $x > 0$ and $t > 0$, with initial conditions $u(x,0) = \varphi(x)$ and $u_t(x,0) = 0$ for $x > 0$ and boundary condition $u_x(0,t) = 0$ for $t > 0$.

(c) Solve the wave equation $u_{tt} = c^2 u_{xx}$ for $x > 0$ and $t > 0$ with initial conditions $u(x,0) = 0$ and $u_t(x,0) = \varphi(x)$ for $x > 0$ and boundary conditions $u_x(0,t) = 0$ for $t > 0$.

With the boundary condition $u_x(x,0) = 0$ in each part of this problem, we need to extend the initial data as an even function of x. So let

$$F(x) = \begin{cases} \varphi(x) & \text{for } x > 0 \\ \varphi(-x) & \text{for } x < 0 \end{cases} = \begin{cases} 1 & \text{if } -L < x < L \\ 0 & \text{if } |x| > L \end{cases}$$

(a) For the heat equation, we have

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)F(y)\,dy = \frac{1}{\sqrt{4\pi kt}}\int_{-L}^{L} e^{-\frac{(x-y)^2}{4kt}}\,dy$$

Now let $p = (x-y)/\sqrt{4kt}$ so that $dp = -dy/\sqrt{4kt}$. Then

$$u(x,t) = \frac{1}{\sqrt{\pi}}\int_{(x+L)/\sqrt{4kt}}^{(x-L)/\sqrt{4kt}} e^{-p^2}(-dp) = \frac{1}{\sqrt{\pi}}\int_{(x-L)/\sqrt{4kt}}^{(x+L)/\sqrt{4kt}} e^{-p^2}\,dp = \frac{1}{2}\left(\mathcal{E}\mathrm{rf}\left(\frac{x+L}{4kt}\right) - \mathcal{E}\mathrm{rf}\left(\frac{x-L}{4kt}\right)\right)$$

(b) Since $u_t(x,0) = 0$, we only have the first part of d'Alembert's solution, so

$$u(x,t) = \frac{1}{2}(F(x+ct) + F(x-ct))$$

Since we're only solving when $x > 0$ and $t > 0$, there are three possibilities:

1. $x - ct < -L$ (in which case $x + ct > L$), so $F(x+ct) = F(x-ct) = 0$ and we have $u(x,t) = 0$.
2. $-L < x - ct < L$, which splits into two sub-cases, depending on the value of $x + ct$:
   (a) $x + ct < L$, so $F(x+ct) = 1$ and $F(x-ct) = 1$, and we have $u(x,t) = \frac{1}{2}(1+1) = 1$.
   (b) $x + ct > L$, so $F(x+ct) = 0$ and $F(x-ct) = 1$ and we have $u(x,t) = \frac{1}{2}(0+1) = \frac{1}{2}$.
3. $x - ct > L$ (in which case $x + ct > L$), so $F(x+ct) = F(x-ct) = 0$ and we have $u(x,t) = 0$.

Therefore

$$u(x,t) = \begin{cases} 0 & \text{if } x - ct < -L \text{ or } x - ct > L \\ 1 & \text{if } -L < x - ct < L \text{ and } x + ct < L \\ \frac{1}{2} & \text{if } -L < x - ct < L \text{ and } x + ct > L \end{cases}$$

It's instructive to visualize the shape of the string for various values of t.

- When $t = 0$ (the initial data) the function $u(x,0)$ is 1 for x between 0 and L and drops off to zero at L.
- For small positive values of t ($0 < t < L/c$), the shape of the string has a segment where $u(x,0) = 1$ for $0 < x < L - ct$ that appears to be moving to the left and off the end of the string, then a segment of height $\frac{1}{2}$ for x between $L - ct$ and $L + ct$, which seems to be expanding, and then $u = 0$ for larger x.
- Right at time $t = L/c$, the segment of height 1 has just entirely moved off the end of the string, and there is a segment of height $\frac{1}{2}$ that stretches from $x = 0$ to $x = 2L$, beyond which $u = 0$
- For time $t > L/c$. there is a rectangular "bump" in the string of height $\frac{1}{2}$ that stretches from $x = ct - L$ to $x = ct + L$ and appears to be moving to the right. For x outside this bump, $u = 0$.

(c) This time we have $u(x,0) = 0$ so we only have the second part of d'Alembert's solution:

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} F(s)\,ds.$$

As in the previous case, there are three possibilities

1. $x - ct < -L$ (in which case $x + ct > L$) and the interval of integration contains the interval where $F = 1$, so the integral is equal to $2L$ and $u(x,t) = L/c$ in this case.
2. $-L < x - ct < L$, which again splits into two sub-cases, depending on the value of $x + ct$:
   (a) $x + ct < L$, so we have that
   $$-L < x - ct < x + ct < L$$
   so the entire interval of integration is contained in the interval where $F = 1$, so the integral is $2ct$ and so $u = t$
   (b) $x + ct > L$, so we have that
   $$-L < x - ct < L < x + ct$$
   so only the part of the interval of integration between $x - ct$ and L is contained in the interval where $F = 1$, so the value of the integral is $L - x + ct$ and the value of u at such a point is $\dfrac{L - x + ct}{2c}$.
3. $x - ct > L$ (in which case $x + ct > L$) so the interval of integration does not intersect the interval where $F = 1$, so the integral and $u(x,t)$ are both zero.

Therefore
$$u(x,t) = \begin{cases} 2L & \text{if } x - ct < -L \\ t & \text{if } -L < x - ct < L \text{ and } x + ct < L \\ \dfrac{L - x + ct}{2c} & \text{if } -L < x - ct < L \text{ and } x + ct > L \\ 0 & \text{if } x - ct > L \end{cases}$$

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3. Find all functions $\varphi(x)$ for which the "traveling wave" $u(x,t) = \varphi(x - ct)$ is a solution of the *heat* equation $u_t = ku_{xx}$. Repeat for the "heat wave" traveling in the other direction: $u(x,t) = \varphi(x + ct)$.

If $u(x,t) = \varphi(x - ct)$, then $u_t = -c\varphi'(x - ct)$ and $u_x x = \varphi''(x - ct)$. Substitute into the heat equation and get that
$$u_t - ku_{xx} = -c\varphi' - k\varphi'' = 0$$
in other words
$$k\varphi'' + c\varphi' = 0.$$
the solutions of the characteristic equation $kr^2 + cr = 0$ are $r = 0$ and $r = -c/k$. Therefore
$$\varphi(x) = c_1 + c_2 e^{-cx/k} \qquad \text{if } c \neq 0$$
and
$$\varphi(x) = c_1 + c_2 x \qquad \text{if } c = 0$$
and the corresponding solutions of the heat equation are
$$u(x,t) = c_1 + c_2 e^{-c(x-ct)/k} \qquad \text{and} \qquad u(x,t) = c_1 + c_2 x.$$

If $u(x,t) = \varphi(x+ct)$, then $u_t = c\varphi'(x-ct)$ and $u_x x = \varphi''(x-ct)$. Substitute into the heat equation and get that

$$u_t - ku_{xx} = c\varphi' - k\varphi'' = 0$$

in other words

$$k\varphi'' - c\varphi' = 0.$$

the solutions of the characteristic equation $kr^2 - cr = 0$ are $r = 0$ and $r = c/k$. Therefore

$$\varphi = c_1 + c_2 e^{cx/k} \qquad \text{if } c \neq 0$$

and

$$\varphi(x) = c_1 + c_2 x \qquad \text{if } c = 0$$

and the corresponding solutions of the heat equation are

$$u(x,t) = c_1 + c_2 e^{c(x+ct)/k} \qquad \text{and} \qquad u(x,t) = c_1 + c_2 x.$$

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4. Solve the modified (damped) wave equation:

$$u_{tt} + 2u_t + u = u_{xx}$$

on the whole line with initial data $u(x,0) = 2xe^{-x^2}$ and $u_t(x,0) = 1$. (*Hint*: Consider $w(x,t) = e^t u(x,t)$)

Following the hint, let $w(x,t) = e^t u(x,t)$. Then

$$w_{xx}(x,t) = e^t u_{xx}(x,t)$$

and

$$w_t(x,t) = e^t u(x,t) + e^t u_x(x,t)$$

(by the product rule) and

$$w_{tt}(x,t) = [e^t u(x,t) + u_t(x,t)] + [e^t u_t(x,t) + e^t u_{tt}(x,t)] = e^t u_{tt}(x,t) + 2e^t u_t(x,t) + e^t u(x,t)$$

(by the product rule again). Therefore w satisfies the regular wave equation $w_{tt} = w_{xx}$ (with $c = 1$) if and only if u satisfies the equation in the problem. Since $e^0 = 1$, the initial data for w are

$$w(x,0) = e^0 u(x,0) = 2xe^{-x^2} \quad \text{and} \quad w_t(x,0) = e^0 u(x,0) + e^0 u_t(x,0) = 2xe^{-x^2} + 1.$$

Therefore

$$\begin{aligned} w(x,t) &= \frac{1}{2}\Big(2(x+t)e^{-(x+t)^2} + 2(x-t)e^{-(x-t)^2}\Big) + \frac{1}{2}\int_{x-t}^{x+t} 2se^{-s^2} + 1\, ds. \\ &= (x+t)e^{-(x+t)^2} + (x-t)e^{-(x-t)^2} + \frac{1}{2}\Big(s - e^{-s^2}\Big)\Big|_{s=x-t}^{s=x+t} \\ &= (x+t)e^{-(x+t)^2} + (x-t)e^{-(x-t)^2} + t + \frac{1}{2}\Big(e^{-(x-t)^2} - e^{-(x+t)^2}\Big) \end{aligned}$$

Therefore

$$u(x,t) = e^{-t}w(x,t) = e^{-t}\left[(x+t)e^{-(x+t)^2} + (x-t)e^{-(x-t)^2} + t + \frac{1}{2}\Big(e^{-(x-t)^2} - e^{-(x+t)^2}\Big)\right]$$

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5. The purpose of this problem is to show that a solution of the heat equation on a finite interval with constant boundary conditions will converge as $t \to \infty$ to the equilibrium solution. We need a preliminary result, though, and the first few parts of this problem lead up to that (it might seem irrelevant at first, but bear with it). In any part of the problem, you may assume that the results of the previous parts are true, even if you skip them. In particular, you might want to start with part (f), and use the result of part (e) when needed.

(a) Let $p(x) = ax^2 + bx + c$ be a quadratic polynomial with the property that $p(x) \geq 0$ for all x (so p does not have two distinct real roots). Explain why this implies that $b^2 \leq 4ac$.

(b) Now consider two (differentiable) functions $u(x)$ and $v(x)$ defined on the interval $0 \leq x \leq L$. Then define

$$p(\lambda) = \int_0^L (|u(x)| + \lambda|v(x)|)^2 \, dx.$$

Once you multiply this out, you can see that $p(\lambda)$ is a quadratic polynomial. Use this together with part (a) to show that

$$\left(\int_0^L |u(x)v(x)| \, dx\right)^2 \leq \int_0^L u(x)^2 \, dx \int_0^L v(x)^2 \, dx.$$

(c) Next, suppose that $f(x)$ is a differentiable function on the interval $0 \leq x \leq L$, and suppose further that $f(0) = f(L) = 0$ (Dirichlet!). Explain why

$$|f(x)| \leq \int_0^L |f'(x)| \, dx$$

for all x between 0 and L.

(d) The quantity on the right side of part (c) is just a number (i.e., a constant). So square both sides, and integrate them from 0 to L and explain why

$$\int_0^L (f(x))^2 \, dx \leq L\left(\int_0^L |f'(x)| \, dx\right)^2$$

(e) Now use the result of part (b) (with $u(x) = f'(x)$ and $v(x) = 1$) to explain why

$$\int_0^L (f(x))^2 \, dx \leq L^2 \int_0^L (f'(x))^2 \, dx.$$

Here comes the PDE part: Let $u(x,t)$ be the solution of the heat equation

$$u_t = ku_{xx} \quad \text{for } t > 0 \text{ and } 0 < x < L,$$

with

$$\text{initial condition } u(x,0) = \varphi(x) \text{ for } 0 < x < L$$

and

$$\text{(Dirichlet) boundary conditions } u(0,t) = u(L,t) = 0 \text{ for } t > 0.$$

As usual, define the energy function $E(t)$ via

$$E(t) = \frac{1}{2}\int_0^L (u(x,t))^2\,dx.$$

so that $E(t) \geq 0$ for all $t \geq 0$ and $E(0) = \displaystyle\int_0^L (\varphi(x))^2\,dx$.

(f) Use the differential equation, integration by parts, and the result of part (e) (with $f(x) = u(x,t)$ for a fixed value of t) to show that

$$\frac{dE}{dt} \leq -\frac{2k}{L^2}E$$

(g) Now divide both sides by E (which we know is positive), integrate from 0 to t and exponentiate both sides to conclude that

$$0 \leq E(t) \leq E(0)e^{-2kt/L^2}.$$

(h) Conclude that $E(t) \to 0$ as $t \to \infty$, and hence $u(x,t) \to 0$ as $t \to \infty$.

(i) Now consider the initial/boundary value problem

$$u_t = ku_{xx} \qquad \text{for } t > 0 \text{ and } 0 < x < L,$$

with

$$\text{initial condition } u(x,0) = \varphi(x) \text{ for } 0 < x < L$$

and

$$\text{(Dirichlet) boundary conditions } u(0,t) = p \text{ and } u(L,t) = q \text{ for } t > 0.$$

Find the equilibrium solution (the time-independent solution of the PDE that satisfies the boundary conditions, ignoring the initial conditions), and then prove that no matter what the initial condition $\varphi(x)$ is, the solution $u(x,t)$ will approach the equilibrium solution as $t \to \infty$.

(a) The roots of the polynomial are

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

by the quadratic formula. The roots will be real and distinct (so the parabolic graph of $p(x)$ will dip below the x-axis) if the quantity under the radical is positive, i.e., if $b^2 - 4ac > 0$. If either the roots are real and equal or are complex, then we must have $b^2 - 4ac \leq 0$, or $b^2 \leq 4ac$.

(b) Multiply out:

$$p(\lambda) = \int_0^L (|u(x)| + \lambda|v(x)|)^2\,dx = \int_0^L u(x)^2\,dx + 2\lambda\int_0^L |u(x)v(x)|\,dx + \lambda^2\int_0^L v(x)^2\,dx.$$

We can think of this as a quadratic polynomial in λ: $p(x) = a\lambda^2 + b\lambda + c$ with coefficients

$$a = \int_0^L v(x)^2\,dx \qquad b = 2\int_0^L u(x)v(x)\,dx \qquad c = \int_0^L u(x)^2\,dx.$$

But $p(\lambda) \geq 0$ for all λ, since the original definition of $p(\lambda)$ was the integral of a square, which is either positive or zero. So from part (a) we get

$$b^2 \leq 4ac \quad \text{in other words} \quad \left(2\int_0^L |u(x)v(x)|\,dx\right)^2 \leq 4\int_0^L v(x)^2\,dx \int_0^L u(x)^2\,dx.$$

Divide both sides by 4 to obtain the result, which is usually called the *Cauchy-Schwarz inequality.*

(c) By the fundamental theorem of calculus:

$$\int_0^x f'(t)\,dt = f(x) - f(0) = f(x)$$

(because $f(0) = 0$) so taking absolute values we get

$$|f(x)| = \left|\int_0^x f'(t)\,dt\right| \leq \int_0^x |f'(t)|\,dt \leq \int_0^L |f'(t)|\,dt = \int_0^L |f'(x)|\,dx$$

provided $0 < x < L$ (the second inequality because integrating a non-negative function over a bigger interval can only increase the result).

(d) Just to make it clear that we understand that the result on the right side of part (c) is a number, let

$$K = \int_0^L |f'(x)|\,dx.$$

Then part (c) says, for $0 \leq x \leq L$,

$$|f(x)| \leq K \qquad \text{or} \qquad f(x)^2 \leq K^2.$$

Therefore

$$\int_0^L (f(x))^2\,dx \leq \int_0^L K^2\,dx = K^2 L = L\left(\int_0^L |f'(x)|\,dx\right)^2$$

(e) Start from the end of part (d):

$$\int_0^L (f(x))^2\,dx \leq \int_0^L K^2\,dx = K^2 L = L\left(\int_0^L |f'(x)|\,dx\right)^2$$

$$= L\left(\int_0^L |f'(x)|(1)\,dx\right)^2 \leq L\int_0^L (f'(x))^2\,dx \int_0^L 1^2\,dx = L^2\int_0^L (f'(x))^2\,dx.$$

This is (a weak form of) the (one-dimensional) *Poincaré inequality.*

(f) First calculate:

$$\frac{dE}{dt} = \frac{\partial}{\partial t}\frac{1}{2}\int_0^L u(x,t)^2\,dx = \int_0^L u(x,t)u_t(x,t)\,dx = k\int_0^L u(x,t)u_{xx}(x,t)\,dx$$

since u satisfies the heat equation. Now integrate by parts (with $f = u$, $dg = u_{xx}\,dx$ so $df = u_x\,dx$ and $g = u_x$) and get

$$\frac{dE}{dt} = ku(x,t)u_x(x,t)\Big|_{x=0}^{L} - k\int_0^L (u_x(x,t))^2\,dx = -k\int_0^L (u_x(x,t))^2\,dx$$

since $u(0,t) = u(L,t) = 0$. Now use the result of part (e) , which says (after dividing by L^2) that

$$\int_0^L (u_x(x,t))^2\,dx \geq \frac{1}{L^2}\int_0^L (u(x,t))^2\,dx$$

or, on negating both sides (and switching greater than to less than!)

$$-\int_0^L (u_x(x,t))^2\,dx \leq -\frac{1}{L^2}\int_0^L (u(x,t))^2\,dx$$

Therefore

$$\frac{dE}{dt} = -k\int_0^L (u_x(x,t))^2\,dx \leq -\frac{k}{L^2}\int_0^L (u(x,t))^2\,dx = -\frac{2k}{L^2}E$$

(g) Dividing both sides by E doesn't change the sense of the inequality since $E > 0$, and gives

$$\frac{1}{E}\frac{dE}{dt} \leq -\frac{2k}{L^2}.$$

Integrate both sides from 0 to t and get

$$\ln E(t) - \ln E(0) \leq -\frac{2kt}{L^2}$$

Exponentiating both sides doesn't change the sense of the inequality since e^x is a monotonically increasing function, and gives:

$$\frac{E(t)}{E(0)} \leq e^{-2kt/L^2} \qquad \text{or} \qquad E(t) \leq E(0)e^{-2kt/L^2}$$

This kind of reasoning (going from a differential inequality to an inequality about the function) is a special case of *Gronwall's inequality.* And of course we know that $E(t) \geq 0$ so we have

$$0 \leq E(t) \leq E(0)e^{-2kt/L^2}.$$

(h) Since $E(t)$ is "trapped" between 0 and e^{-2kt/L^2} and since the latter approaches 0 as $t \to \infty$, the "squeeze theorem" from elementary calculus implies that

$$\lim_{t\to\infty} E(t) = 0.$$

In other words,

$$\lim_{t\to\infty} (u(x,t))^2\,dx = 0 \quad \text{which implies} \quad \lim_{t\to\infty} u(x,t) = 0$$

for all (well, at least "almost all") x.

(i) The equilibrium solution satisfies $u_{xx} = 0$ since u is independent of t. Therefore the equilibrium solution is the linear function of x satisfying $u(0) = p$ and $u(L) = q$, i.e,

$$u(x) = p + \frac{q-p}{L}x$$

If we let

$$v(x,t) = u(x,t) - p - \frac{q-p}{L}x,$$

then v will satisfy the heat equation

$$v_t = kv_{xx} \quad \text{for } t > 0 \text{ and } 0 < x < L,$$

with

$$\text{initial condition } v(x,0) = \varphi(x) - p - \frac{q-p}{L}x \text{ for } 0 < x < L$$

and

$$\text{boundary conditions } v(0,t) = 0 \text{ and } v(L,t) = 0 \text{ for } t > 0.$$

So v is a function to which the conclusions of parts (e),(f),(g) and (h) apply, so we have

$$\lim_{t\to\infty} v(x,t) = 0 \quad \text{for } 0 < x < L$$

and therefore

$$\lim_{t\to\infty} u(x,t) = \lim_{t\to\infty}\left(v(x,t) + p + \frac{q-p}{L}x\right) = p + \frac{q-p}{L}x.$$

So u approaches the equilibrium solution as $t \to \infty$.