Math 425 / AMCS 525 **Dr. DeTurck**
Midterm Exam 2 **March 31, 2016**

There are four problems on this test. You may use your book and your notes during this exam. Do as much of it as you can during the class period, and turn your work in at the end. But take the sheet with the problems home with you, and you may (re)work any problems you like and turn them in on Tuesday for additional credit.

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1. Here are two problems that *are* as straightforward as they look. You shouldn't have to do any integrals to solve them.

(a) Solve the heat equation:

$$u_t = 2u_{xx} \quad \text{for } 0 < x < 1,\ t > 0$$

with the initial condition

$$u(x,0) = \sin\frac{\pi}{2}x$$

and the boundary conditions

$$u(0,t) = 0 \quad \text{and} \quad u_x(1,t) = 0$$

(so homogeneous Dirichlet on the left and homogeneous Neumann on the right).

(b) Solve the Laplace equation

$$u_{xx} + u_{yy} = 0 \qquad \text{for } 0 < x < 2,\ 0 < y < 1$$

with boundary conditions

$$u(x,0) = u(x,1) = 0, \qquad u(0,y) = 0,\ u(2,y) = \sin(4\pi y).$$

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2. Solve the heat equation:

$$u_t = 2u_{xx} \quad \text{for } 0 < x < 1,\ t > 0$$

with the initial condition

$$u(x,0) = 2x - x^2$$

and the boundary conditions

$$u(0,t) = 1 \quad \text{and} \quad u_x(1,t) = 0$$

(so *inhomogeneous* Dirichlet on the left and homogeneous Neumann on the right).

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3. Consider the eigenvalue problem

$$X'' + \lambda X = 0, \qquad X(0) = 0, \qquad hX(1) + X'(1) = 0,$$

where h is a real parameter.

(a) Show that for any (real) value of h, all the eigenvalues are real.

(b) Show that if $h > 0$ then there are only positive eigenvalues (use "Green's first identity" or, more prosaically, integration by parts).

(c) If $h < 0$, what additional restriction on h is necessary to ensure that the smallest positive eigenvalue is less than $(\pi/2)^2$? (*Hint*: For this one, find the non-zero solutions of the differential equation and the boundary condition at $x = 0$ that would correspond to a positive value of λ, and then write [and interpret] the equation λ would have to satisfy in order to fulfill the boundary condition at $x = 1$. Pay particular attention to what happens near $\lambda = 0$.)

(d) What value(s) of h will lead to the eigenvalue $\lambda = 0$?

(e) If $h < 0$, what additional restriction on h will guarantee the existence of a (at least one) negative eigenvalue? Is there a negative value of h for which there is more than one negative eigenvalue? (*Hint*: Approach this part as you did in part (c), except now λ is negative; it's probably easier to use hyperbolic functions than exponentials).

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4. (a) Find the Fourier series (on the interval $-\pi < x < \pi$) of the function $f(x) = \cos \alpha x$ where α is *not* an integer. (*Note*: Even though $\cos \alpha x$ is not periodic with period 2π, it's still *even*. That should save you at least one integral.)

(b) Replace α by z in your answer to (a), so now we're going to think of z as the variable (and x as a parameter). By choosing a particular value of x (not z!), obtain the series expansion:

$$\csc \pi z = \frac{1}{\pi z} + \frac{2z}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{z^2 - n^2}.$$

Choose another value of x and obtain the series expansion:

$$\cot \pi z = \frac{1}{\pi z} + \frac{2z}{\pi} \sum_{n=1}^{\infty} \frac{1}{z^2 - n^2}.$$

For what values of z do the series converge? (These two series are like partial-fraction decompositions for functions with infinitely many singularities.)

(c) For $0 < x < 1$, integrate the series in (b) for $\cot \pi z$ from $z = 0$ to $z = x$ (assume term-by-term integration is possible, but pay attention to show that what you're doing makes sense near $z = 0$), to get

$$\ln\left(\frac{\sin \pi x}{\pi x}\right) = \sum_{n=1}^{\infty} \ln\left(1 - \frac{x^2}{n^2}\right).$$

(d) Exponentiate both sides in (c) to get

$$\sin \pi x = \pi x \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2}\right).$$

(This is an infinite product expansion – like the factorization of a polynomial with one factor for each root, except that the sine function has infinitely many roots.)

Extra credit: Look up the definition of convergence of an infinite product. For what values of x does the infinite product converge? For what values of x does it converge to $\sin \pi x$? (So far, we've only proved this for $0 < x < 1$.)