Math 425 / AMCS 525 **Dr. DeTurck**
Midterm Exam 2 **March 31, 2016**

There are four problems on this test. You may use your book and your notes during this exam. Do as much of it as you can during the class period, and turn your work in at the end. But take the sheet with the problems home with you, and you may (re)work any problems you like and turn them in on Tuesday for additional credit.

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1. Here are two problems that *are* as straightforward as they look. You shouldn't have to do any integrals to solve them.

(a) Solve the heat equation:

$$u_t = 2u_{xx} \quad \text{for } 0 < x < 1,\ t > 0$$

with the initial condition

$$u(x,0) = \sin\frac{\pi}{2}x$$

and the boundary conditions

$$u(0,t) = 0 \quad \text{and} \quad u_x(1,t) = 0$$

(so homogeneous Dirichlet on the left and homogeneous Neumann on the right).

(b) Solve the Laplace equation

$$u_{xx} + u_{yy} = 0 \qquad \text{for } 0 < x < 2,\ 0 < y < 1$$

with boundary conditions

$$u(x,0) = u(x,1) = 0, \qquad u(0,y) = 0,\ u(2,y) = \sin(4\pi y).$$

(a) The eigenfunctions for the boundary conditions $X(0) = 0$ and $X'(1) = 0$ are $\sin(n+\frac{1}{2})\pi x$. Since the initial data is one of the eigenfunctions (for $n = 0$), it is its own Fourier series, so the answer will be the single term

$$u(x,t) = e^{-\frac{1}{2}\pi^2 t}\sin\left(\frac{\pi x}{2}\right).$$

(b) Since the boundary conditions are zero on the segments where $y = 0$ and $y = 1$, the Y eigenfunctions will be $\sin n\pi y$, and since the X boundary conditions are $X(0) = 0$ and inhomogeneous at $x = 2$, the X factors in the separated solutions will be $X = \sinh n\pi x$. Again the inhomogeneous boundary condition is one of the Y eigenfunctions, so there will be a single term in the answer (the one with $n = 4$), namely

$$u(x,t) = \frac{\sinh 4\pi x}{\sinh 8\pi}\sin 4\pi y.$$

2. Solve the heat equation:

$$u_t = 2u_{xx} \quad \text{for } 0 < x < 1,\ t > 0$$

with the initial condition

$$u(x,0) = 2x - x^2$$

and the boundary conditions

$$u(0,t) = 1 \quad \text{and} \quad u_x(1,t) = 0$$

(so *inhomogeneous* Dirichlet on the left and homogeneous Neumann on the right).

First, let $v(x,t) = u(x,t) - 1$ so that v satisfies the heat equation $v_t = 2v_{xx}$ together with boundary conditions

$$v(0,t) = 0 \quad \text{and} \quad v_x(1,t) = 0$$

and the initial condition

$$v(x,0) = 2x - x^2 - 1 = -(x-1)^2.$$

The boundary conditions for the separated solutions for v are $X(0) = 0$ and $X'(1) = 0$, so $X(x) = \sin(n + \frac{1}{2})x$, and

$$v(x,t) = \sum_{n=0}^{\infty} A_n e^{-2(n+\frac{1}{2})^2\pi^2 t} \sin((n + \tfrac{1}{2})\pi x)$$

where

$$A_n = \frac{\langle -(x-1)^2, \sin(n+\frac{1}{2})\pi x\rangle}{\langle \sin(n+\frac{1}{2})\pi x, \sin(n+\frac{1}{2})\pi x\rangle}$$

Now

$$\left\langle \sin(n + \frac{1}{2})\pi x, \sin(n + \frac{1}{2})\pi x \right\rangle = \int_0^1 \sin^2(n + \tfrac{1}{2})\pi x \, dx = \int_0^1 \frac{1}{2}(1 - \cos(2n+1)\pi x)\, dx$$

$$= \frac{1}{2}x - \frac{1}{2(2n+1)\pi} \sin(2n+1)\pi x \Big|_0^1 = \frac{1}{2}$$

and

$$\left\langle -(x-1)^2, \sin(n + \frac{1}{2})\pi x \right\rangle = \int_0^1 -(x-1)^2 \sin(n + \tfrac{1}{2})\pi x \, dx$$

$$= \frac{(x-1)^2}{(n+\frac{1}{2})\pi} \cos(n+\tfrac{1}{2})\pi x - \frac{2(x-1)}{(n+\frac{1}{2})^2\pi^2} \sin(n+\tfrac{1}{2})\pi x - \frac{2}{(n+\frac{1}{2})^3\pi^3} \cos(n+\tfrac{1}{2})\pi x \Big|_0^1$$

$$= \frac{-1}{(n+\frac{1}{2})\pi} + \frac{2}{(n+\frac{1}{2})^3\pi^3}$$

So

$$v(x,t) = \sum_{n=0}^{\infty} 2\left[\frac{2}{(n+\frac{1}{2})^3\pi^3} - \frac{1}{(n+\frac{1}{2})\pi}\right] e^{-2(n+\frac{1}{2})^2\pi^2 t} \sin((n+\tfrac{1}{2})\pi x)$$

and

$$u(x,t) = 1 + \sum_{n=0}^{\infty} \left[\frac{4}{(n+\frac{1}{2})^3\pi^3} - \frac{2}{(n+\frac{1}{2})\pi}\right] e^{-2(n+\frac{1}{2})^2\pi^2 t} \sin((n+\tfrac{1}{2})\pi x)$$

3. Consider the eigenvalue problem

$$X'' + \lambda X = 0, \qquad X(0) = 0, \qquad hX(1) + X'(1) = 0,$$

where h is a real parameter.

(a) Show that for any (real) value of h, all the eigenvalues are real.

(b) Show that if $h > 0$ then there are only positive eigenvalues (use "Green's first identity" or, more prosaically, integration by parts).

(c) If $h < 0$, what additional restriction on h is necessary to ensure that the smallest positive eigenvalue is less than $(\pi/2)^2$? (*Hint*: For this one, find the non-zero solutions of the differential equation and the boundary condition at $x = 0$ that would correspond to a positive value of λ, and then write [and interpret] the equation λ would have to satisfy in order to fulfill the boundary condition at $x = 1$. Pay particular attention to what happens near $\lambda = 0$.)

(d) What value(s) of h will lead to the eigenvalue $\lambda = 0$?

(e) If $h < 0$, what additional restriction on h will guarantee the existence of a (at least one) negative eigenvalue? Is there a negative value of h for which there is more than one negative eigenvalue? (*Hint*: Approach this part as you did in part (c), except now λ is negative; it's probably easier to use hyperbolic functions than exponentials).

(a) If f and g are functions that satisfy the boundary conditions, then $f'(1) = -hf(1)$ and $g'(1) = -hg(1)$ and so

$$f'g - g'f\Big|_0^1 = -hf(1)g(1) + hg(1)f(1) + hf(0)g(0) - hg(0)f(0) = 0$$

Therefore this is a "symmetric" eigenvalue problem and the theorem (Theorem 2 on page 121 of the text) guarantees that all the eigenvalues are real and Theorem 1 on page 120 says that eigenfunctions corresponding to distinct eigenvalues are orthogonal.

(b) Suppose $h > 0$. Then (since $X'' + \lambda X = 0$)

$$\begin{aligned}\lambda \langle X, X \rangle = -\langle X'', X \rangle = -\int_0^1 X''(x)X(x)\,dx &= -\,XX'\Big|_0^1 + \int_0^1 (X'(x))^2\,dx \\ &= X(0)X'(0) - X(1)X'(1) + \int_0^1 (X'(x))^2\,dx \\ &= h(X(1))^2 + \int_0^1 (X'(x))^2\,dx \geq 0\end{aligned}$$

and so $\lambda \geq 0$ (since $\langle X, X \rangle = \|X\| > 0$). And part (d) below tells us that 0 is not an eigenvalue unless $h = -1$, so $\lambda > 0$.

(c) If $\lambda > 0$ say $\lambda = \beta^2$. Then $X(x) = c \sin \beta x$ (because $X(0) = 0$). Then the condition $X'(1) = -hX(1)$ tells us that $h \sin \beta + \beta \cos \beta = 0$. In other words,

$$\frac{\beta}{-h} = \tan \beta.$$

In order for there to be a root of this equation that occurs to the left of the first horizontal asymptote of $\tan \beta$ (namely $\beta = \frac{1}{2}\pi$), we need the slope of $-\beta/h$ to be greater than 1, in other words we need $-1/h > 1$, which means that $-1 < h$ in other words $0 > h > -1$.
(d) If $\lambda = 0$, then $X'' = 0$ so $X = Ax$ (since $X(0) = 0$). Then $hX(1) + X'(1) = hA + A = 0$ implies that $h = -1$.

(e) If there is a negative eigenvalue λ, then let $\lambda = -\beta^2$ so that $X(x) = A \sinh \beta x$ (so that $X(0) = 0$). Then

$$hX(1) + X'(1) = A(h \sinh \beta + \beta \cosh \beta).$$

If this is zero then

$$\frac{\beta}{-h} = \tanh \beta$$

Now $\tanh \beta$ has derivative equal to 1 when $\beta = 0$ and is concave down, so if a line through the origin is going to intersect it then the line must have positive slope less than 1:

$$0 < \frac{1}{-h} < 1$$

which implies that $-1 > h$. There can't be more than one negative eigenvalue since $\tanh \beta$ is concave down so a line can't intersect it more than twice (and one time is at $\beta = 0$).

4. (a) Find the Fourier series (on the interval $-\pi < x < \pi$) of the function $f(x) = \cos \alpha x$ where α is *not* an integer. (*Note*: Even though $\cos \alpha x$ is not periodic with period 2π, it's still *even*. That should save you at least one integral.)

(b) Replace α by z in your answer to (a), so now we're going to think of z as the variable (and x as a parameter). By choosing a particular value of x (not z!), obtain the series expansion:

$$\csc \pi z = \frac{1}{\pi z} + \frac{2z}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{z^2 - n^2}.$$

Choose another value of x and obtain the series expansion:

$$\cot \pi z = \frac{1}{\pi z} + \frac{2z}{\pi} \sum_{n=1}^{\infty} \frac{1}{z^2 - n^2}.$$

For what values of z do the series converge? (These two series are like partial-fraction decompositions for functions with infinitely many singularities.)

(c) For $0 < x < 1$, integrate the series in (b) for $\cot \pi z$ from $z = 0$ to $z = x$ (assume term-by-term integration is possible, but pay attention to show that what you're doing makes sense near $z = 0$), to get

$$\ln\left(\frac{\sin \pi x}{\pi x}\right) = \sum_{n=1}^{\infty} \ln\left(1 - \frac{x^2}{n^2}\right).$$

(d) Exponentiate both sides in (c) to get

$$\sin \pi x = \pi x \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2}\right).$$

(This is an infinite product expansion – like the factorization of a polynomial with one factor for each root, except that the sine function has infinitely many roots.)

Extra credit: Look up the definition of convergence of an infinite product. For what values of x does the infinite product converge? For what values of x does it converge to $\sin \pi x$? (So far, we've only proved this for $0 < x < 1$.)

(a) Since $\cos\alpha x$ is even, we have

$$\cos\alpha x = A_0 + \sum_{n=1}^{\infty} A_n \cos nx$$

where

$$A_0 = \frac{\langle\cos\alpha x, 1\rangle}{\langle 1, 1\rangle} = \frac{1}{2\pi\alpha}\sin\alpha x\Big|_{-\pi}^{\pi} = \frac{\sin\alpha\pi}{\alpha\pi}$$

and

$$\begin{aligned} A_n &= \frac{\langle\cos\alpha x, \cos nx\rangle}{\langle\cos nx, \cos nx\rangle} = \frac{1}{\pi}\int_{-\pi}^{\pi}\cos\alpha x\cos nx\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi}\cos(\alpha+n)x + \cos(\alpha-n)x\,dx \\ &= \frac{1}{\pi}\left(\frac{1}{\alpha+n}\sin(\alpha+n)\pi + \frac{1}{k-n}\sin(\alpha-n)\pi\right) \\ &= \frac{(-1)^n\sin\pi\alpha}{\pi}\left(\frac{1}{\alpha+n} + \frac{1}{\alpha-n}\right) = \frac{(-1)^n\sin\pi\alpha}{\pi}\frac{2\alpha}{\alpha^2-n^2} \end{aligned}$$

Therefore

$$\cos\alpha x = \frac{\sin\pi\alpha}{\pi\alpha} + \sum_{n=1}^{\infty}\frac{(-1)^n}{\pi}\sin\pi\alpha\frac{2\alpha}{\alpha^2-n^2}\cos nx$$

(b) Set $\alpha = z$ and obtain

$$\cos xz = \frac{\sin\pi z}{\pi z} + \sum_{n=1}^{\infty}\frac{(-1)^n}{\pi}\frac{2z\sin\pi z}{z^2-n^2}\cos nx = \sin\pi z\left(\frac{1}{\pi z} + \frac{2z}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{z^2-n^2}\cos nx\right)$$

Put $x = 0$ in this and get

$$1 = \sin\pi z\left(\frac{1}{\pi z} + \frac{2z}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{z^2-n^2}\right)$$

Divide both sides by $\sin\pi z$ and conclude

$$\csc\pi z = \frac{1}{\pi z} + \frac{2z}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{z^2-n^2}$$

Next, put $x = \pi$ and get

$$\cos\pi z = \sin\pi z\left(\frac{1}{\pi z} + \frac{2z}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n(-1)^n}{z^2-n^2}\right)$$

which is equivalent to

$$\cot\pi z = \frac{1}{\pi z} + \frac{2z}{\pi}\sum_{n=1}^{\infty}\frac{1}{z^2-n^2}$$

These series for $\csc \pi z$ and $\cot \pi z$ will converge (absolutely) for all $z \notin \mathbb{Z}$, by (limit) comparison with the convergent series $\sum \frac{1}{n^2}$, since

$$\lim_{n\to\infty} \frac{\left|\dfrac{1}{z^2-n^2}\right|}{\dfrac{1}{n^2}} = 1$$

provided z is not an integer.

(c) In order to have the integration make sense, we move the first term to the left and write:

$$\int_0^x \cot \pi z - \frac{1}{\pi z}\, dz = \int_0^x \frac{2z}{\pi} \sum_{n=1}^{\infty} \frac{1}{z^2-n^2}\, dz$$

The integral on the left is improper at $z = 0$ so we calculate:

$$\begin{aligned}
\int_0^x \cot \pi z - \frac{1}{\pi z}\, dz &= \lim_{a\to 0+} \int_a^x \frac{\cos \pi z}{\sin \pi z} - \frac{1}{\pi z}\, dz \\
&= \lim_{a\to 0+} \frac{1}{\pi} \left(\ln \sin \pi z - \ln \pi z\right)\Big|_a^{\mathbf{x}} \\
&= \lim_{a\to 0+} \frac{1}{\pi} \ln \frac{\sin \pi z}{\pi z}\Big|_a^x \\
&= \frac{1}{\pi} \ln \frac{\sin \pi x}{\pi x}
\end{aligned}$$

We move the $2z$ and the integral into the sum on the right to get

$$\begin{aligned}
\int_0^x \frac{2z}{\pi} \sum_{n=1}^{\infty} \frac{1}{z^2-n^2}\, dz &= \frac{1}{\pi} \sum_{n=1}^{\infty} - \int_0^x \frac{2z}{n^2-z^2}\, dz \\
&= \frac{1}{\pi} \sum_{n=1}^{\infty} \ln(n^2-z^2)\Big|_0^x \\
&= \frac{1}{\pi} \sum_{n=1}^{\infty} \ln(n^2-x^2) - \ln n^2 \\
&= \frac{1}{\pi} \sum_{n=1}^{\infty} \ln\left(1 - \frac{x^2}{n^2}\right)
\end{aligned}$$

Therefore (upon multiplication by π)

$$\ln \frac{\sin \pi x}{\pi x} = \sum_{n=1}^{\infty} \ln\left(1 - \frac{x^2}{n^2}\right)$$

(d) Exponentiating this last equation gives

$$\frac{\sin \pi x}{\pi x} = \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2}\right)$$

so multiply by πx to get

$$\sin \pi x = \pi x \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2}\right)$$

Extra credit: An infinite product of positive numbers converges if and only if the series of logarithms of the factors converges. And the series at the end of part (c) converges absolutely by limit comparison with the convergent series $\sum \frac{1}{n^2}$, since (using L'Hôpital's rule)

$$\lim_{n\to\infty} \frac{\left|\ln\left(1 - \frac{x^2}{n^2}\right)\right|}{\frac{1}{n^2}} = \left|\lim_{n\to\infty} \frac{\frac{2x^2}{n^3}}{\left(1 - \frac{x^2}{n^2}\right)\frac{2}{n^3}}\right| = x^2$$

We know the product converges to $\sin \pi x$ for $0 \le x \le 1$ (both sides are clearly zero at $x = 0$ and $x = 1$). But both sides are odd functions of x, so the product converges to $\sin \pi x$ for $-1 \le x \le 1$. The basic definition of convergence for an infinite product is via the convergence of the sequence of "partial products":

$$\prod_{n=1}^{\infty} p_n = \lim_{N\to\infty} \prod_{n=1}^{N} p_n$$

So we'll show that the infinite product

$$P(x) = \pi x \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2}\right)$$

defines a periodic function of x with period 2, i.e., that $P(x+2) = P(x)$ which will prove that the product converges to $\sin \pi x$ for all x. We have

$$\begin{aligned}
P(x+2) &= \pi(x+2)\prod_{n=1}^{\infty}\left(1-\frac{(x+2)^2}{n^2}\right) = \lim_{N\to\infty}\pi(x+2)\prod_{n=1}^{N}\left(1-\frac{(x+2)^2}{n^2}\right) \\
&= \lim_{N\to\infty}\pi(x+2)\prod_{n=1}^{N}\frac{1}{n^2}(n-(x+2))(n+(x+2)) \\
&= \lim_{N\to\infty}\pi(x+2)\left(\prod_{n=1}^{N}\frac{1}{n^2}\right)\left(\prod_{n=1}^{N}-(x-(n-2))(x-(-n-2))\right) \\
&= \lim_{N\to\infty}(-1)^N\pi(x+2)\left(\prod_{n=1}^{N}\frac{1}{n^2}\right)\left(\prod_{n=-N-2}^{-3}(x-n)\right)\left(\prod_{n=-1}^{N-2}(x-n)\right) \\
&= \lim_{N\to\infty}(-1)^N\pi\left(\prod_{n=1}^{N}\frac{1}{n^2}\right)\left(\prod_{n=-N-2}^{N-2}(x-n)\right) \\
&= \lim_{N\to\infty}(-1)^N\pi\frac{(x-(-N-2))(x-(-N-1))}{(x-(N-1))(x-N)}\left(\prod_{n=1}^{N}\frac{1}{n^2}\right)\left(\prod_{n=-N}^{N}(x-n)\right) \\
&= \lim_{N\to\infty}(-1)^N\pi\frac{(x+N+2)(x+N+1)}{(x-N+1)(x-N)}\left(\prod_{n=1}^{N}\frac{1}{n^2}\right)x\prod_{n=1}^{N}(x^2-n^2) \\
&= \lim_{N\to\infty}\frac{(x+N+2)(x+N+1)}{(x-N+1)(x-N)}\pi x\prod_{n=1}^{N}\left(1-\frac{x^2}{n^2}\right) \\
&= \pi x\prod_{n=1}^{\infty}\left(1-\frac{x^2}{n^2}\right) = P(x)
\end{aligned}$$

Therefore $P(x)$ is periodic and so equals $\sin\pi x$ for all x.