Math 425 / AMCS 525 **Dr. DeTurck**
Final exam **Due Wednesday, May 4, 2016**

Heat equation in 3D

Section 9.4, page 253, problem 1 Let's look for a polynomial solution. Since $u_t = k\Delta u$ and $u(x,y,z,0) = xy^2z$, note that $\Delta(xy^2z) = 2xz$. So let's try

$$u = xy^2z + 2ktxz$$

Then

$$u_t = 2kxz, \quad \text{and} \quad \Delta u = 2xz$$

so u satisfies the heat equation, and

$$u(x,y,z,0) = xy^2z.$$

So this is the solution of the problem.

Section 9.4, page 253, problem 3 Let

$$f(x,y,z) = \begin{cases} \varphi(x,y,z) & \text{if } z > 0 \\ \varphi(x,y,-z) & \text{if } z < 0 \end{cases}$$

be the even extension of φ. Then

$$\begin{aligned} u(x,y,z,t) &= \frac{1}{(4\pi kt)^{3/2}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-((x-x')^2+(y-y')^2+(z-z')^2)/(4kt)} f(x',y',z')\,dx'dy'dz' \\ &= \frac{1}{(4\pi kt)^{3/2}} \int_{0}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-((x-x')^2+(y-y')^2+(z-z')^2)/(4kt)} \varphi(x',y',z')\,dx'dy'dz' \\ &\qquad + \frac{1}{(4\pi kt)^{3/2}} \int_{-\infty}^{0}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-((x-x')^2+(y-y')^2+(z-z')^2)/(4kt)} \varphi(x',y',-z')\,dx'dy'dz' \\ &= \frac{2}{(4\pi kt)^{3/2}} \int_{0}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-((x-x')^2+(y-y')^2+(z-z')^2)/(4kt)} \varphi(x',y',z')\,dx'dy'dz' \end{aligned}$$

Schrödinger and Hermite

Section 9.4, page 253, problem 4 We'll interpret "from scratch" as meaning "from the differential equation" (16) on page 252. If

$$w'' - 2xw' + (\lambda - 1)w = 0 \quad \text{and we let} \quad w = \sum_{n=0}^{\infty} a_n x^n$$

so that

$$2xw' = \sum_{n=1}^{\infty} 2a_n n x^n \quad \text{and} \quad w'' = \sum_{n=2}^{\infty} a_n n(n-1)x^{n-2}$$

then the differential equation becomes

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=0}^{\infty}(2n-\lambda+1)a_n x^n = 0$$

or, letting $m = n-2$ (or $n = m+2$) in the first sum, and $m = n$ in the second:

$$\sum_{m=0}^{\infty}\Big[(m+2)(m+1)a_{m+2} - (2m-\lambda+1)a_m\Big]x^m = 0.$$

This implies the recursion relation

$$(m+2)(m+1)a_{m+2} = (2m-\lambda+1)a_m \quad \text{for } m \geq 0$$

We write $\lambda = 2n+1$, and use the normalization for $H_n(x)$ that the coefficient of x^n in H_n is 2^n.

So if $n = 0$ then $\lambda = 1$ and we let $a_0 = 1$ and $a_1 = 0$, then $2a_2 = 0$ so all $a_m = 0$ for $m \geq 1$ and $w(x) = 1$, so $H_0(x) = 1$.

If $n = 1$ then $\lambda = 3$ and we let $a_0 = 0$ and $a_1 = 1$, in which case $a_3 = 0$ so all $a_m = 0$ for $m \geq 2$ and $w(x) = x$, so $H_1(x) = 2x$.

If $n = 2$ then $\lambda = 3$ and we let $a_0 = 1$ and $a_1 = 0$, in which case (for $m = 0$) $2a_2 = -4a_0$ and (for $m = 2$) $12a_4 = 0$, and so $a_m = 0$ for $m \geq 3$ and $w(x) = 1 - 2x^2$, so $H_2(x) = 4x^2 - 2$.

Finally, if $n = 3$ then $\lambda = 7$ and we let $a_0 = 0$ and $a_1 = 1$, in which case (for $m = 1$) $6a_3 = -4a_1$ and (for $m = 3$) a_5=0, and so $a_2 = 0$ and $a_m = 0$ for all $m \geq 4$ and $w(x) = x - \frac{2}{3}x^3$, so $H_3(x) = 8x^3 - 12$.

Section 9.4, page 253, problem 5 This is ever-so-slightly different from the homework problem, which was about h_n. We must show that if

$$H_k(x) = (-1)^k e^{x^2}\frac{d^k}{dx^k}e^{-x^2}$$

then $H_k'' - 2xH_k' + 2kH_k = 0$. First, calculate:

$$H_k' = (-1)^k 2xe^{x^2}\frac{d^k}{dx^k}e^{-x^2} + (-1)^k\frac{d^{k+1}}{dx^{k+1}}e^{-x^2}$$

and

$$H_k'' = (-1)^k 2e^{x^2}\frac{d^k}{dx^k}e^{-x^2} + (-1)^k 4x^2e^{x^2}\frac{d^k}{dx^k}e^{-x^2} + (-1)^k 4xe^{x^2}\frac{d^{k+1}}{dx^{k+1}}e^{-x^2} + (-1)^k e^{x^2}\frac{d^{k+2}}{dx^{k+2}}e^{-x^2}$$

so right off, the first term of $-2xH_k'$ will cancel the second term of H_k'', the second term of $-2xH_k'$ will add to the third term of H_k'' to make the coefficient $-2+4=2$, and the first term of H_k'' will add to $2kH_k$ to make the coefficient $2k+2$. So

$$\begin{aligned} H_k'' - 2xH_k' + 2kH_k &= (-1)^k(2k+2)e^{x^2}\frac{d^k}{dx^k}e^{-x^2} + (-1)^k 2xe^{x^2}\frac{d^{k+1}}{dx^{k+1}}e^{-x^2} + (-1)^k e^{x^2}\frac{d^{k+2}}{dx^{k+2}}e^{-x^2} \\ &= (-1)^k e^{x^2}\left((2k+2)\frac{d^k}{dx^k}e^{-x^2} + 2x\frac{d^{k+1}}{dx^{k+1}}e^{-x^2} + \frac{d^{k+2}}{dx^{k+2}}e^{-x^2}\right) \end{aligned}$$

Next, we'll use the fact that for any function $f(x)$,

$$\frac{d^n}{dx^n}(xf(x)) = x\frac{d^n}{dx^n}f(x) + n\frac{d^{n-1}}{dx^{n-1}}f(x)$$

with $n = k+1$, to write

$$\frac{d^{k+2}}{dx^{k+2}}e^{-x^2} = \frac{d^{k+1}}{dx^{k+1}}\left(\frac{d}{dx}e^{-x^2}\right) = -2\frac{d^{k+1}}{dx^{k+1}}\left(xe^{-x^2}\right) = -2x\frac{d^{k+1}}{dx^{k+1}}\left(e^{-x^2}\right) - 2(k+1)\frac{d^k}{dx^k}\left(e^{-x^2}\right)$$

But this shows that the last term in the expression for $H_k'' - 2xH_k' + 2kH_k$ precisely cancels the first two, so

$$H_k'' - 2xH_k' + 2kH_k = 0$$

as claimed.

We also have to show that $H_k(x)$ as defined in the problem is in fact a polynomial. Since from the recurrence relation for the coefficients of the solution of the differential equation (from the preceding problem) we can conclude that (up to a multiplicative constant) there is only one polynomial solution of the equation, we can then conclude that the formula given in this problem yields a constant multiple of $H_k(x)$ — in fact, it is equal to $H_k(x)$ since we'll also show that the leading term is $2^k x^k$. We proceed by induction. Certainly

$$(-1)^0 e^{x^2}\frac{d^0}{dx^0}e^{-x^2} = e^{x^2}e^{-x^2} = 1 = H_0(x)$$

so the result is true for $k = 0$. So assume that

$$H_k(x) = (-1)^k e^{x^2}\frac{d^k}{dx^k}e^{-x^2}$$

and calculate

$$(-1)^{k+1}e^{x^2}\frac{d^{k+1}}{dx^{k+1}}e^{-x^2} = -e^{x^2}\frac{d}{dx}\left(e^{-x^2}(-1)^k e^{x^2}\frac{d^k}{dx^k}e^{-x^2}\right) = -e^{x^2}\frac{d}{dx}\left(e^{-x^2}H_k(x)\right)$$

But we're assuming that H_k is a polynomial with leading term $2^k x^k$, and the right-hand side is equal to

$$2xH_k(x) - H_k'(x),$$

which is clearly a polynomial with leading term $2^{k+1}x^{k+1}$, so we're justified in calling it $H_{k+1}(x)$.

Section 9.4, page 253, problem 6 This one is actually a homework problem, but anyhow:

We know that $v = H_k(x)e^{-x^2/2}$ satisfies the differential equation

$$v'' + (\lambda - x^2)v = 0$$

with $\lambda = 2k+1$. So we have

$$
\begin{aligned}
(2k+1)\left\langle H_k(x)e^{-x^2/2}, H_\ell(x)e^{-x^2/2}\right\rangle &= \left\langle (2k+1)H_k(x)e^{-x^2/2}, H_\ell(x)e^{-x^2/2}\right\rangle \\
&= \left\langle x^2(H_k(x)e^{-x^2/2}) - (H_k(x)e^{-x^2/2})'', H_\ell(x)e^{-x^2/2}\right\rangle \\
&= \int_{-\infty}^{\infty} \left(x^2(H_k(x)e^{-x^2/2}) - (H_k(x)e^{-x^2/2})''\right)\left(H_\ell(x)e^{-x^2/2}\right)\,dx \\
&= \int_{-\infty}^{\infty} \left(H_k(x)e^{-x^2/2}\right)\left(x^2(H_\ell(x)e^{-x^2/2}) - (H_\ell(x)e^{-x^2/2})''\right)\,dx \\
&= \left\langle H_k(x)e^{-x^2/2}, x^2(H_\ell(x)e^{-x^2/2}) - (H_\ell(x)e^{-x^2/2})''\right\rangle H_\ell(x)e^{-x^2/2} \\
&= (2\ell+1)\left\langle H_k(x)e^{-x^2/2}, H_\ell(x)e^{-x^2/2}\right\rangle
\end{aligned}
$$

Since $k \neq \ell$ this implies that $\left\langle H_k(x)e^{-x^2/2}, H_\ell(x)e^{-x^2/2}\right\rangle = 0$, in other words, that

$$0 = \int_{-\infty}^{\infty} \left(H_k(x)e^{-x^2/2}\right)\left(H_\ell(x)e^{-x^2/2}\right)\,dx = \int_{-\infty}^{\infty} H_k(x)H_\ell(x)e^{-x^2}\,dx.$$

Spherical coordinates and Legendre

Sectiom 10.3, page 277, problem 3 I apologize for this one, since given the definition in the text on page 275, the identity in the problem can't possibly be correct, since it depends on m only via $|m|$ and $e^{im\varphi}$. But see

`http://mathworld.wolfram.com/SphericalHarmonic.html`

for definitions of the spherical harmonics for which the identity is true.

Section 10.3, page 277, problem 5 The problem is to solve $\mathbf{u}_t = k\Delta u$ on the ball of radius a, with $u = B$ on the boundary of the ball and $u = C$ when $t = 0$. Since everything about this is invariant under rotations, we'll seek a radial solution and write the equation as

$$u_t = k\left(u_{rr} + \frac{2u_r}{r}\right).$$

Then we'll let $v(r,t) = u(r,t) - B$, so that the problem for v is

$$v_t = k\left(v_{rr} + \frac{2v_r}{r}\right) \qquad \text{with} \quad v(r,0) = C - B \qquad V(a,t) = 0 \qquad V(0,t) \text{ finite}$$

Write $v(r,t) = R(r)T(t)$ then the usual separation of variables gives

$$\frac{T'}{kT} = \frac{rR'' + 2R'}{rR} = -\lambda$$

where we expect $\lambda > 0$. The equation for R (after dividing by r is

$$R'' + \frac{2}{r}R' + \lambda R = 0.$$

Let's try a change of variables of the form $R(r) = r^p w(r)$ for some p. Then $R' = r^p w' + pr^{p-1}w$ and $R'' = r^p w'' + 2pr^{p-1}w' + p(p-1)r^{p-2}w$. So the equation would become

$$r^p w'' + 2pr^{p-1}w' + p(p-1)r^{p-2}w + 2r^{p-1}w' + 2pr^{p-2}w + \lambda r^p w = 0$$

or

$$r^p w'' + \left(2pr^{p-1} + 2r^{p-1}\right) w' + \left(p(p-1)r^{p-2} + 2pr^{p-2} + \lambda r^p\right) w = 0$$

If we put $p = -1$ this becomes

$$r^{-1}w'' + \lambda r^{-1}w = 0$$

or

$$w'' + \lambda w = 0.$$

The boundary conditions for w are $w(a) = 0$ and $w(0) = 0$ (since we want $R(0)$ to be finite and $R(r) = w(r)/r$. This tells us that

$$w(r) = \sin\frac{n\pi r}{a} \quad \text{and} \quad \lambda = \frac{n^2\pi^2}{a^2}.$$

Therefore

$$R(r) = \frac{1}{r}\sin\frac{n\pi r}{a} \quad \text{for the eigenvalue} \quad \lambda = \frac{n^2\pi^2}{a^2} \quad n = 1, 2, \ldots$$

The corresponding solutions for T of $T' = -k\lambda T$ are

$$T(t) = e^{-n^2\pi^2 kt/a^2}$$

and so

$$v(r,t) = \sum_{n=1}^{\infty} A_n e^{-n^2\pi^2 kt/a^2}\frac{1}{r}\sin\frac{n\pi r}{a}.$$

Since we want $v(r, 0) = C - B$, we must choose A_n to be the Fourier sine series coefficients of the function $(C - B)r$, so

$$A_n = \frac{2}{a}\int_0^a (C{-}B)r\sin\frac{n\pi r}{a}\,dr = \frac{2(C-B)}{a}\left(-\frac{ar}{n\pi}\cos\frac{n\pi r}{a} + \frac{a^2}{n^2\pi^2}\sin\frac{n\pi r}{a}\Big|_0^a\right) = (-1)^{n+1}\frac{2(C-B)a}{n\pi}$$

Putting this into the formula for v and adding the B back to get u gives the answer

$$u(r,t) = B + \frac{2(C-B)}{\pi r}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}e^{-n^2\pi^2 kt/a^2}\sin\frac{n\pi r}{a}.$$

Section 10.6, page 293, problem 6 We want to solve $\Delta u = 0$ on the ball of radius a with $u = \cos^2\theta$ on the boundary. Keeping in mind that in the book's notation, θ is the (co-)elevation angle rather than the azimuthal one, so the solution should be axially symmetric around the z-axis. Using the observation (on page 277) that the solid spherical harmonics are polynomials, this means that we're looking for a polynomial that depends only on z and on $r^2 = x^2 + y^2 + z^2$. And since $z = r\cos\theta$, the boundary condition says that $u = z^2/a^2$ when $r = a$.

A basis for the space of harmonic polynomials of degree 2 or less that depend only on r and z is $\{h_0, h_2\}$, where

$$h_0(r,z) = 1 \qquad \text{and} \qquad h_2(r,z) = 3z^2 - r^2$$

so our solution has to be of the form $u(r,z) = a_0 + a_2(3z^2 - r^2)$. When $r = a$ this becomes

$$u(a,z) = a_0 + a_2(3z^2 - a^2).$$

For this to equal z^2/a^2, we need $a_2 = 1/(3a^2)$ and $a_0 = \frac{1}{3}$. So

$$u(r,z) = \frac{z^2}{a^2} - \frac{r^2}{3a^2} + \frac{1}{3} = \frac{a^2 - x^2 - y^2 + 2z^2}{3a^2}$$

Fourier transforms

Section 12.3, page 348, problem 6 (a) First, start with the Fourier inversion formula

$$f(x) = \mathcal{F}^{-1}\left[\widehat{f}(\omega)\right](x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{f}(\omega)e^{ix\omega}\,d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} \widehat{f}(\omega)e^{ix\omega}\,d\omega$$

since f is band-limited (i,e, $\widehat{f}(\omega) = 0$ for $|\omega| > \pi$). For $x = n$ an integer, we have

$$f(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \widehat{f}(\omega)e^{in\omega}\,d\omega$$

and this is the $(-n)$th coefficient of the complex Fourier series for the function $\widehat{f}(\omega)$ on the interval $-\pi < \omega < \pi$. Therefore, on this interval,

$$\widehat{f}(\omega) = \sum_{n=-\infty}^{\infty} f(n)e^{-in\omega}$$

Therefore,

$$\begin{aligned} f(x) &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sum_{n=-\infty}^{\infty} f(n)e^{-in\omega}\right)e^{ix\omega}\,d\omega = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} f(n)\int_{-\pi}^{\pi} e^{i(x-n)\omega}\,d\omega \\ &= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} f(n)\left(\frac{e^{i(x-n)\omega}}{i(x-n)}\right)\Bigg|_{\omega=-\pi}^{\pi} = \frac{1}{\pi}\sum_{n=-\infty}^{\infty}\frac{f(n)}{x-n}\left(\frac{e^{i\pi(x-n)} - e^{-i\pi(x-n)}}{2i}\right) \\ &= \sum_{n=-\infty}^{\infty} f(n)\frac{\sin[\pi(x-n)]}{\pi(x-n)} \end{aligned}$$

which is what we were trying to prove.

(b) If $\widehat{f}(\omega) = 1$ for $-\pi < \omega < \pi$ and zero otherwise, then

$$f(x) = \mathcal{F}^{-1}\left[\widehat{f}(\omega)\right](x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ix\omega}\,d\omega = \frac{1}{2\pi}\left(\frac{e^{i\pi x} - e^{-i\pi x}}{ix}\right) = \frac{1}{\pi x}\left(\frac{e^{i\pi x} - e^{-i\pi x}}{2i}\right) = \frac{\sin \pi x}{\pi x}$$

Now for $n \neq 0$, we have that $f(n) = 0$ since $\sin \pi n = 0$. For $n = 0$ we take the limiting value and declare

$$f(0) = \lim_{x \to 0} \frac{\sin \pi x}{\pi x} = 1.$$

Therefore the equation reads simply

$$f(x) = \frac{\sin \pi x}{\pi x}$$

which is true.

Section 12.3, page 348, problem 7 (a) Since $f(x) = 0$ for large values of $|x|$, there is no issue about the convergence of

$$\sum_{n=-\infty}^{\infty} f(x + 2\pi n)$$

since the sum is actually finite. Now (letting $m = n + 1$)

$$g(x + 2\pi) = \sum_{n=-\infty}^{\infty} f(x + 2\pi + 2\pi n) = \sum_{n=-\infty}^{\infty} f(x + 2\pi(n + 1)) = \sum_{m=-\infty}^{\infty} f(x + 2\pi m) = g(x)$$

so $g(x)$ is periodic with period 2π.

(b) Let c_m be the mth Fourier coefficient of $g(x)$. Then

$$\begin{aligned} c_m &= \frac{1}{2\pi} \int_{-\pi}^{\pi} g(x) e^{-imx}\, dx = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \int_{-\pi}^{\pi} f(x + 2\pi n) e^{-imx}\, dx \\ &= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \int_{2\pi n - \pi}^{2\pi n + \pi} f(y) e^{-im(y - 2\pi n)}\, dy = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \int_{2\pi n - \pi}^{2\pi n + \pi} f(y) e^{-imy}\, dy \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(y) e^{-imy}\, dy = \frac{1}{2\pi} \widehat{f}(m) \end{aligned}$$

So now write

$$g(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} = \sum_{n=-\infty}^{\infty} \frac{1}{2\pi} \widehat{f}(n) e^{inx}$$

But also

$$g(x) = \sum_{n=-\infty}^{\infty} f(x + 2\pi n)$$

so putting $x = 0$ we conclude

$$\sum_{n=-\infty}^{\infty} f(2\pi n) = \sum_{n=-\infty}^{\infty} \frac{1}{2\pi} \widehat{f}(n)$$

which is Poisson's summation formula.

Section 12.3, page 348, problem 9 We take the Fourier transform of both sides of the equation (in x), noting that $\widehat{\delta} = 1$ and get

$$(\omega^2 + a^2)\widehat{u} = 1$$

and so

$$\widehat{u} = \frac{1}{\omega^2 + a^2}.$$

From the table on page 345, the inverse Fourier transform of the right side is $\dfrac{e^{-a|x|}}{2a}$. So the (general) solution of the differential equation is

$$u(x) = \frac{e^{-a|x|}}{2a} + c_1 e^{-ax} + c_2 e^{ax}$$

(The last two exponential terms do not come from the FT method because the Fourier transforms of those functions do not exist. So when we wrote $\widehat{u}$ we tacitly set c_1 and c_2 to zero.)

Electromagnetism

Section 13.1, page 361, problem 3 (a) From Maxwell's equation (II) we have that

$$\frac{\partial \mathbf{B}}{\partial t} = -c\nabla \times \mathbf{E}$$

and since $\mathbf{B} = \nabla \times \mathbf{A}$ we have

$$\frac{\partial}{\partial t}(\nabla \times \mathbf{A}) + c\nabla \times \mathbf{E} = \mathbf{0}$$

or

$$\nabla \times \left(\frac{\partial \mathbf{A}}{\partial t} + c\mathbf{E} \right) = \mathbf{0}$$

or (dividing by c)

$$\nabla \times \left(\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} + \mathbf{E} \right) = \mathbf{0}$$

But since the curl of this vector field is zero, the vector field must be the gradient of a function, which the textbook chooses to call $-u$, so there is a function u such that

$$-\nabla u = \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} + \mathbf{E}$$

(b) Take the divergence of this last identity and obtain

$$-\Delta u = \frac{1}{c}\nabla \cdot \frac{\partial \mathbf{A}}{\partial t} + \nabla \cdot \mathbf{E} = \frac{1}{c}\nabla \cdot \frac{\partial \mathbf{A}}{\partial t} + 4\pi\rho$$

Therefore

$$-\frac{1}{c}\nabla \cdot \frac{\partial \mathbf{A}}{\partial t} - \Delta u = 4\pi\rho$$

which is the first thing we are supposed to show.

Next, start from

$$-\nabla u = \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}$$

and rewrite it as

$$\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} = -\nabla u - \mathbf{E}.$$

Take the derivative of both sides with respect to t, divide by c, and use (I) of Maxwell to get

$$\frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = -\frac{1}{c}\nabla\frac{\partial u}{\partial t} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = -\frac{1}{c}\nabla\frac{\partial u}{\partial t} - \left[\nabla \times \mathbf{B} - \frac{4\pi}{c}\mathbf{J}\right]$$

Now we observe that $\nabla \times \mathbf{B} = \nabla \times \nabla \times \mathbf{A}$ and we have to use the vector identity

$$\nabla \times \nabla \times \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \Delta \mathbf{A}$$

Just for completeness, here's a proof of that:

$$\text{If} \quad \mathbf{A} = \begin{bmatrix} A^1 \\ A^2 \\ A^3 \end{bmatrix} \quad \text{then} \quad \nabla \times \mathbf{A} = \begin{bmatrix} A^3_y - A^2_z \\ A^1_z - A^3_x \\ A^2_x - A^1_y \end{bmatrix}.$$

Consequently,

$$\begin{aligned}
\nabla \times \nabla \times \mathbf{A} &= \begin{bmatrix} A^2_{xy} - A^1_{yy} + A^3_{xz} - A^1_{zz} \\ A^3_{yz} - A^2_{zz} + A^1_{xy} - A^2_{xx} \\ A^1_{xz} - A^3_{xx} + A^2_{yz} - A^3_{yy} \end{bmatrix} \\
&= \begin{bmatrix} A^2_{xy} - A^1_{yy} + A^3_{xz} - A^1_{zz} + A^1_{xx} - A^1_{xx} \\ A^3_{yz} - A^2_{zz} + A^1_{xy} - A^2_{xx} + A^2_{yy} - A^2_{yy} \\ A^1_{xz} - A^3_{xx} + A^2_{yz} - A^3_{yy} + A^3_{zz} - A^3_{zz} \end{bmatrix} \\
&= \begin{bmatrix} (\nabla \cdot \mathbf{A})_x - \Delta A^1 \\ (\nabla \cdot \mathbf{A})_y - \Delta A^2 \\ (\nabla \cdot \mathbf{A})_z - \Delta A^3 \end{bmatrix} = \nabla(\nabla \cdot \mathbf{A}) - \Delta \mathbf{A}.
\end{aligned}$$

Using this identity gives us

$$\frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = -\frac{1}{c}\nabla\frac{\partial u}{\partial t} - \nabla(\nabla \cdot \mathbf{A}) + \Delta \mathbf{A} + \frac{4\pi}{c}\mathbf{J}$$

which we can rearrange as

$$\frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} - \Delta \mathbf{A} + \nabla\left(\nabla \cdot \mathbf{A} + \frac{1}{c}\frac{\partial u}{\partial t}\right) = \frac{4\pi}{c}\mathbf{J}$$

which is what we were supposed to show.

(c) Let $\widetilde{\mathbf{A}} = \mathbf{A} + \nabla\lambda$ abd $\widetilde{u} = u - \frac{1}{c}\frac{\partial \lambda}{\partial t}$. Then

$$\widetilde{\mathbf{B}} = \nabla \times \widetilde{\mathbf{A}} = \nabla \times \mathbf{A} + \nabla\lambda = \nabla \times \mathbf{A} = \mathbf{B}$$

because the curl of a gradient is always zero. Likewise,

$$\begin{aligned}
\widetilde{\mathbf{E}} &= -\nabla\widetilde{u} - \frac{1}{c}\frac{\partial \widetilde{\mathbf{A}}}{\partial t} = -\nabla u + \frac{1}{c}\nabla\frac{\partial \lambda}{\partial t} - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} - \frac{1}{c}\nabla\frac{\partial \lambda}{\partial t} \\
&= -\nabla u - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} = \mathbf{E}
\end{aligned}$$

Therefore $\widetilde{\mathbf{B}}$ and $\widetilde{\mathbf{E}}$ satisfy Maxwell's equations just as $\mathbf{B}$ and $\mathbf{E}$ do, so the equations in (a) and (b) are true for them as well.

(d) We calculate

$$\nabla \cdot \widetilde{\mathbf{A}} + \frac{1}{c}\frac{\partial \widetilde{u}}{\partial t} = \nabla \cdot \mathbf{A} + \Delta\lambda + \frac{1}{c}\left[\frac{\partial u}{\partial t} - \frac{1}{c^2}\frac{\partial^2 \lambda}{\partial t^2}\right]$$

This quantity will be zero if and only if λ satisfies the inhomogeneous wave equation

$$\frac{\partial^2 \lambda}{\partial t^2} - c^2\Delta\lambda = c^2\nabla \cdot \mathbf{A} + c\frac{\partial u}{\partial t}.$$

Since we know that the inhomogeneous wave equation has solutions, we can make the original quantity on the left zero by choosing λ appropriately.

(e) If we assume that λ is chosen as indicated in part (d), then

$$\nabla \cdot \frac{\partial \widetilde{\mathbf{A}}}{\partial t} = -\frac{1}{c}\frac{\partial^2 \widetilde{u}}{\partial t^2}$$

so that the first equation from part (b) becomes

$$\frac{1}{c^2}\frac{\partial^2 \widetilde{u}}{\partial t^2} - \Delta\widetilde{u} = 4\pi\rho$$

and the second equation becomes

$$\frac{1}{c^2}\frac{\partial^2 \widetilde{\mathbf{A}}}{\partial t} - \Delta\widetilde{\mathbf{A}} + \nabla\left(\frac{1}{c}\frac{\partial \widetilde{u}}{\partial t} - \frac{1}{c}\frac{\partial \widetilde{u}}{\partial t}\right) = \frac{1}{c^2}\frac{\partial^2 \widetilde{\mathbf{A}}}{\partial t} - \Delta\widetilde{\mathbf{A}} = \frac{4\pi}{c}\mathbf{J}$$

which is what we wanted to show.

Section 13.1, page 361, problem 4 The book gives the proof that $\mathbf{E}$ satisfies the wave equation. As for $\mathbf{B}$, we have that

$$\frac{\partial \mathbf{B}}{\partial t} = -c\nabla \times \mathbf{E},$$

so

$$\frac{\partial^2 \mathbf{B}}{\partial t^2} = -c\nabla \times \frac{\partial \mathbf{E}}{\partial t} = -c\nabla \times (c\nabla \times \mathbf{B}) = -c^2\nabla \times \nabla \times \mathbf{B} = -c^2\Big(\nabla(\nabla \cdot \mathbf{B}) - \Delta\mathbf{B}\Big) = c^2\Delta\mathbf{B}$$

using that vector identity again, and Maxwell (IV). So $\mathbf{B}$ also satisfies the wave equation.

Shock waves

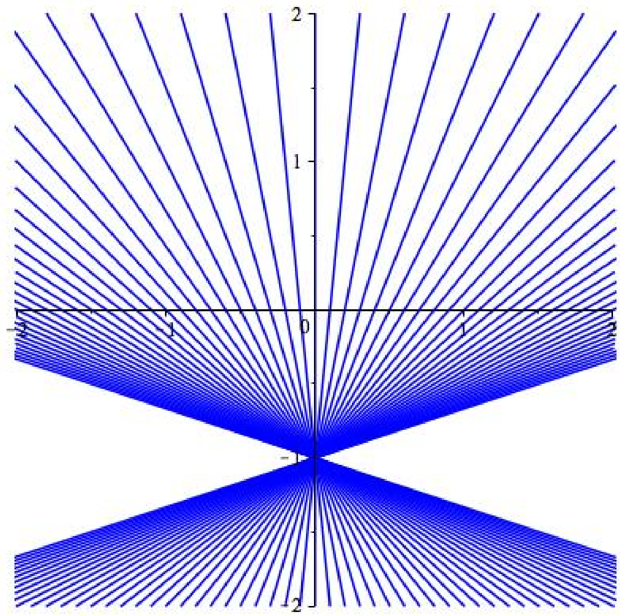
Section 14.1, page 389, problem 3 Solve this by the standard method for first-order equations, namely, consider the system

$$\frac{dt}{ds} = 1 \qquad \frac{dx}{ds} = u \qquad \frac{du}{ds} = 0$$

with initial conditions

$$t(0) = 0 \qquad x(0) = x_0 \qquad u(0) = x_0$$

The solution is $t = s$, $x = x_0(s+1)$, $u = x_0$, which tells us that $u = \dfrac{x}{t+1}$. Clearly this is valid only for $t. > -1$. Here are some of the characteristics:



Section 14.1, page 389, problem 5 Again, we convert the problem to a system of ODEs:

$$\frac{dt}{ds} = 1 \qquad \frac{dx}{ds} = u^2 \quad \frac{du}{ds} = 0$$

with initial conditions

$$t(0) = 0 \qquad x(0) = x_0 \qquad u(0) = 2 + x_0$$

so $t = s$, $u = 2 + x_0$, and $x = (2 + x_0)^2 t + x_0$, so that $x = u^2 t + u - 2$, or in other words

$$u = \frac{-1 \pm \sqrt{1 + 8t + 4xt}}{2t}$$

We must choose the + sign so that u satisfies the initial condition $u(0) = 2 + x$, so

$$u = \frac{-1 + \sqrt{1 + 8t + 4xt}}{2t}$$

Section 14.1, page 389, problem 7 We'll take the hint and let $y = x^2$. We'll take the cases of $x > 0$ and $x < 0$ separately. For $x > 0$, $x = \sqrt{y}$ and so

$$\frac{\partial u}{\partial x} = \frac{dy}{dx}\frac{\partial u}{\partial y} = 2\sqrt{y}\,\frac{\partial u}{\partial y}.$$

In the variables y and t, the problem becomes

$$\sqrt{y}\,u_t + 2\sqrt{y}\,uu_y = 0 \qquad u(y, 0) = \sqrt{y}$$

or

$$u_t + 2uu_y = 0 \qquad u(y, 0) = \sqrt{y}$$

We convert to a system of ODEs:

$$\frac{dt}{ds} = 1 \qquad \frac{dy}{ds} = 2u \qquad \frac{du}{ds} = 0$$

with initial conditions

$$t(0) = 0 \qquad y(0) = y_0 \qquad u(0) = \sqrt{y_0}$$

so $t = s$, $u = \sqrt{y_0}$, and then $y = y_0 + 2\sqrt{y_0}\, s = u^2 + 2ut$. We conclude that

$$u = \frac{-2t \pm \sqrt{4t^2 + 4y}}{2} = -t \pm \sqrt{t^2 + y} = -t \pm \sqrt{t^2 + x^2}$$

We have to take the + sign so that $u(x, 0) = x$ and so

$$u = -t + \sqrt{t^2 + x^2}$$

when $x > 0$.

For $x < 0$, we still have $y = x^2$ but now $x = -\sqrt{y}$. So in the variables y and t, the problem becomes

$$-\sqrt{y}\, u_t - 2\sqrt{y}\, uu_y = 0 \qquad u(y, 0) = -\sqrt{y}$$

or

$$u_t + 2uu_y = 0 \qquad u(y, 0) = -\sqrt{y}$$

The system of ODEs is the same but the initial conditions are

$$t(0) = 0 \qquad y(0) = y_0 \qquad u(0) = -\sqrt{y_0}$$

so $t = s$, $u = -\sqrt{y_0}$, and then $y = y_0 - 2\sqrt{y_0}\, s = u^2 + 2ut$. We conclude that

$$u = \frac{-2t \pm \sqrt{4t^2 + 4y}}{2} = -t \pm \sqrt{t^2 + y} = -t \pm \sqrt{t^2 + x^2}$$

But now we have to take the − sign so that $u(x, 0) = x$ and so

$$u = -t - \sqrt{t^2 + x^2}$$

when $x < 0$.

Calculus of variations

Section 14.3, page 400, problem 1 We're trying to minimize

$$\int_a^b \sqrt{1 + p^2}\, dx$$

where $p = y'$. The Euler-Lagrange equations in this case become

$$0 = \frac{d}{dx} \frac{p}{\sqrt{1 + p^2}}$$

so

$$\frac{y'}{\sqrt{1 + y'^2}} = c_1$$

This implies that y' is a constant, or that the graph is a straight line. These are the only stationary curves, so they must be minima (since we know there's no maximum and that there *must* be a minimum).

Section 14.3, page 400, problem 2 We're trying to minimize

$$\int_0^1 \sqrt{1+y'^2}\,dx$$

subject to the constraint

$$A = \int_0^1 y\,dx$$

So we consider the functional

$$\int_0^1 \sqrt{1+y'^2} + \lambda y\,dx$$

where λ is a Lagrange multiplier. So $F(x,y,p) = \sqrt{1+p^2} + \lambda y$ and the Euler-Lagrange equations become

$$\lambda = \frac{d}{dx}\frac{y'}{\sqrt{1+y'^2}}$$

which immediately integrate (with respect to x) to

$$\lambda x + c = \frac{y'}{\sqrt{1+y'^2}}.$$

Square both sides and rearrange to get

$$(\lambda x + c)^2 = y'^2\Big(1-(\lambda x+c)^2\Big)$$

or

$$y' = \frac{\lambda x + c}{\sqrt{1-(\lambda x+c)^2}}$$

To integrate the right side, let $u = 1-(\lambda x+c)^2$, then $du = -2\lambda(\lambda x+c)\,dx$ and we get

$$y = \int \frac{\lambda x + c}{\sqrt{1-(\lambda x+c)^2}} = -\frac{1}{2\lambda}\int u^{-1/2}\,du = -\frac{1}{\lambda}(\sqrt{u}+k) = -\frac{1}{\lambda}\Big(\sqrt{1-(\lambda x+c)^2}+k\Big).$$

A little algebra gives

$$(\lambda x+c)^2 + (\lambda y+k)^2 = 1$$

so the solution curve is an arc of a circle connecting $(0,a)$ and $(1,b)$. Solving for the constants is a real bear, so we'll leave it at that.

Section 14.3, page 400, problem 4 This time we have $F(x,y,p) = p^2 + xy$, so the Euler-Lagrange equations become

$$x = \frac{d}{dx}2p$$

or in other words $2y'' = x$, which implies $y' = \frac{1}{4}x^2 + c$ or $y = \frac{1}{12}x^3 + cx + d$. Now $y(0) = 0$ implies that $d = 0$, and then $y(1) = 1$ implies that $c = \frac{11}{12}$, so

$$y = \frac{1}{12}x^3 + \frac{11}{12}x.$$
