Math 425 / AMCS 525 **Dr. DeTurck**
Assignment 10 **Due Tuesday, April 26, 2016**

Topics for this week — Fourier transforms, using them to solve PDE problems

Tenth Homework Assignment- due Tuesday, April 26

1. Using Fourier transforms, solve the diffusion problem with convection:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} - \gamma u \qquad -\infty < x < \infty$$

$$u(x,0) = f(x)$$

2. Using Fourier transforms, solve:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + c\frac{\partial u}{\partial x} \qquad -\infty < x < \infty$$

$$u(x,0) = f(x)$$

3. Using Fourier transforms, solve:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

for $-\infty < x < \infty$ and $0 < y < H$ with boundary conditions

$$u(x,0) = f_1(x) \quad \text{and} \quad u(x,H) = f_2(x)$$

(assume that f_1 and f_2 are functions of x whose Fourier transforms exist).

4. We discussed the sequence of functions $f_n(x)$, starting from $f_0(x) = e^{-x^2/2}$ and defined inductively by $f_{n+1}(x) = f_n'(x) - xf_n(x)$. Let's change things just a little, just to get rid of pesky minus signs, and define:

$$h_0(x) = e^{-x^2/2}$$

and

$$h_{n+1}(x) = xh_n(x) - h_n'(x) \qquad (*)$$

(so h_n will be $(-1)^n f_n$, and this will save us a lot of $(-1)^n$'s along the way).

(a) Show that h_n is an eigenfunction of the Fourier transform with eigenvalue $(-i)^n\sqrt{2\pi}$, in other words

$$\widehat{h_n}(\omega) = (-i)^n\sqrt{2\pi}h_n(\omega).$$

(b) Explain why h_n is equal to a polynomial of degree n times $e^{-x^2/2}$, say $h_n(x) = H_n(x)e^{-x^2/2}$.

(c) Find the next six of these polynomials, after $H_0(x) = 1$.

(d) Show inductively (or at least give a good argument to convince yourself and others) that

$$xh_n + h_n' = 2nh_{n-1}$$

(if we set $h_{-1}(x) = 0$, this works for $n = 0$, too).

(e) Put the definition (*) of h_n together with part (d) and show that

$$h_n'' - x^2 h_n + (2n+1)h_n = 0.$$

(This shows that h_n is an eigenfunction of the operator $L[f] = f'' - x^2 f$ with eigenvalue $2n+1$).

(f) Using (e), show that h_n is orthogonal to h_m on the whole line if $n \neq m$ with respect to the standard inner product

$$\langle f\,,\, g\rangle = \int_{-\infty}^{\infty} f(x)g(x)\,dx.$$

The differential equation $y'' - x^2 y + \lambda y = 0$ is called Hermite's equation – and its eigenfunctions $h_n(x)$ are called Hermite functions (and the polynomials $H_n(x)$ are called Hermite polynomials. They arise when you use parabolic coordinates ($x = s^2 - t^2$, $y = 2st$) to separate variables in Laplace's equation, as well as in the study of the quantum harmonic oscillator (which we'll talk about in class if there's time).