Math 425 / AMCS 525 **Dr. DeTurck**
Assignment 10 **Due Tuesday, April 26, 2016**

A few hints:

3. Using Fourier transforms, solve:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

for $-\infty < x < \infty$ and $0 < y < H$ with boundary conditions

$$u(x,0) = f_1(x) \quad \text{and} \quad u(x,H) = f_2(x)$$

(assume that f_1 and f_2 are functions of x whose Fourier transforms exist).

On this one, it appears that the best you can do is compute the Fourier transform of the answer (in terms of the Fourier transforms of f_1 and f_2). So you can express the answer as the inverse Fourier transform of some function of ω and y.

4. We discussed the sequence of functions $f_n(x)$, starting from $f_0(x) = e^{-x^2/2}$ and defined inductively by $f_{n+1}(x) = f_n'(x) - xf_n(x)$. Let's change things just a little, just to get rid of pesky minus signs, and define:

$$h_0(x) = e^{-x^2/2}$$

and

$$h_{n+1}(x) = xh_n(x) - h_n'(x) \qquad (*)$$

(so h_n will be $(-1)^n f_n$, and this will save us a lot of $(-1)^n$'s along the way).

(a) Show that h_n is an eigenfunction of the Fourier transform with eigenvalue $(-i)^n\sqrt{2\pi}$, in other words

$$\widehat{h_n}(\omega) = (-i)^n\sqrt{2\pi}h_n(\omega).$$

(b) Explain why h_n is equal to a polynomial of degree n times $e^{-x^2/2}$, say $h_n(x) = H_n(x)e^{-x^2/2}$.

(c) Find the next six of these polynomials, after $H_0(x) = 1$.

(d) Show inductively (or at least give a good argument to convince yourself and others) that

$$xh_n + h_n' = 2nh_{n-1}$$

(if we set $h_{-1}(x) = 0$, this works for $n = 0$, too).

(e) Put the definition (*) of h_n together with part (d) and show that

$$h_n'' - x^2h_n + (2n+1)h_n = 0.$$

(This shows that h_n is an eigenfunction of the operator $L[f] = f'' - x^2 f$ with eigenvalue $2n+1$).

(f) Using (e), show that h_n is orthogonal to h_m on the whole line if $n \neq m$ with respect to the standard inner product

$$\langle f\,,\, g\rangle = \int_{-\infty}^{\infty} f(x)g(x)\,dx.$$

The differential equation $y'' - x^2y + \lambda y = 0$ is called Hermite's equation – and its eigenfunctions $h_n(x)$ are called Hermite functions (and the polynomials $H_n(x)$ are called Hermite polynomials. They arise when you use parabolic coordinates ($x = s^2 - t^2$, $y = 2st$) to separate variables in Laplace's equation, as well as in the study of the quantum harmonic oscillator (which we'll talk about in class if there's time).

Part (d) of this one is *hard*! It will help fist to use the information we have about the h_n, namely

$$h_0(x) = e^{-x^2/2}$$

and

$$h_{n+1}(x) = xh_n(x) - h_n'(x) \tag{$*$}$$

to show that

$$h_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n} \left(e^{-x^2} \right).$$

And once you have this fairly explicit formula for h_n you should be able to prove the new identity in 4(d).