Math 425 / AMCS 525 **Dr. DeTurck**
Assignment 10 **Due Tuesday, April 26, 2016**

1. Using Fourier transforms, solve the diffusion problem with convection:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} - \gamma u \quad -\infty < x < \infty$$

$$u(x,0) = f(x)$$

We take the Fourier transform with respect to x:

$$\widehat{u}_t = -k\omega^2\widehat{u} - \gamma\widehat{u}, \quad \widehat{u}(\omega,0) = \widehat{f}(\omega).$$

Solve the ODE (in the variable t, treating ω as a parameter)

$$\widehat{u}_t + (k\omega^2 + \gamma)\widehat{u} = 0, \quad \widehat{u}(\omega,0) = \widehat{f}(\omega)$$

and get

$$\widehat{u}(\omega,t) = \widehat{f}(\omega)e^{-(k\omega^2+\gamma)t}.$$

Therefore

$$\begin{aligned} u(x,t) &= \mathcal{F}^{-1}\left[\widehat{f}(\omega)e^{-(k\omega^2+\gamma)t}\right](x,t) \\ &= \mathcal{F}^{-1}\left[\left(e^{-\gamma t}\widehat{f}(\omega)\right)\left(e^{-kt\omega^2}\right)\right](x,t) \\ &= \left(e^{-\gamma t}f(x) * \frac{1}{\sqrt{4\pi kt}}e^{-x^2/(4kt)}\right) \\ &= \frac{e^{-\gamma t}}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty} f(y)e^{-(x-y)^2/(4kt)}\,dy. \end{aligned}$$

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2. Using Fourier transforms, solve:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + c\frac{\partial u}{\partial x} \quad -\infty < x < \infty$$

$$u(x,0) = f(x)$$

We take the Fourier transform of both sides of the PDE with respect to x and get

$$\widehat{u}_t = -k\omega^2\widehat{u} + ic\omega\widehat{u}, \quad \widehat{u}(\omega, 0) = \widehat{f}(\omega).$$

The differential equation has become an ordinary differential equation with independent variable t:

$$\widehat{u}_t + (k\omega^2 - ic\omega)\widehat{u} = 0,$$

which has general solution

$$\widehat{u}(\omega, t) = a(\omega)e^{-(k\omega^2 - ic\omega)t}$$

and since

$$\widehat{u}(\omega, 0) = a(\omega) = \widehat{f}(\omega),$$

we have

$$\widehat{u}(\omega, t) = \widehat{f}(\omega)e^{-(k\omega^2 - ic\omega)t}.$$

Therefore,

$$\begin{aligned} u(x,t) &= \mathcal{F}^{-1}\left[\widehat{f}(\omega)e^{-(k\omega^2 - ic\omega)t}\right](x,t). \\ &= \mathcal{F}^{-1}\left[e^{ict\omega}\left(\widehat{f}(\omega)e^{-kt\omega^2}\right)\right](x,t) \\ &= \mathcal{F}^{-1}\left[\widehat{f}(\omega)e^{-kt\omega^2}\right](x+ct,t) \\ &= \frac{1}{2\pi}\left(f(x) * \sqrt{\frac{\pi}{kt}}e^{-x^2/(4kt)}\right)(x+ct,t) \\ &= \frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty} f(y)e^{-(x+ct-y)^2/(4kt)}\,dy \end{aligned}$$

The solution is the same as the solution of the heat equation except translated to the right by ct because of the convection term.

3. Using Fourier transforms, solve:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

for $-\infty < x < \infty$ and $0 < y < H$ with boundary conditions

$$u(x, 0) = f_1(x) \quad \text{and} \quad u(x, H) = f_2(x)$$

(assume that f_1 and f_2 are functions of x whose Fourier transforms exist).

Because the domain is of infinite extent only in the x-direction, we take the Fourier transform with respect to x and obtain the equation

$$\widehat{u}_{yy} - \omega^2\widehat{u} = 0$$

for the function $\widehat{u}(\omega, y)$ with boundary conditions $\widehat{u}(\omega, 0) = \widehat{f}_1(\omega)$ and $\widehat{u}(\omega, H) = \widehat{f}_2(\omega)$. This is an ordinary differential equation (with independent variable y) for the function $\widehat{u}$, and the cleverest way to write the solution is

$$\widehat{u}(\omega, y) = a(\omega)\sinh(y\omega) + b(\omega)\sinh((H-y)\omega)$$

Then for $y = 0$ we have

$$\widehat{u}(\omega, 0) = b(\omega)\sinh(H\omega)$$

and for $y = H$ we have
$$\widehat{u}(\omega, H) = a(\omega)\sinh(H\omega)$$
so we should take
$$b(\omega) = \frac{\widehat{f}_1(\omega)}{\sinh(H\omega)} \quad \text{and} \quad a(\omega) = \frac{\widehat{f}_2(\omega)}{\sinh(H\omega)}.$$
We can conclude that
$$u(x,t) = \mathcal{F}^{-1}\left[\frac{\widehat{f}_2(\omega)\sinh(y\omega)}{\sinh(H\omega)} + \frac{\widehat{f}_1(\omega)\sinh((H-y)\omega)}{\sinh(H\omega)}\right]$$
And that's the best we can do.

4. We discussed the sequence of functions $f_n(x)$, starting from $f_0(x) = e^{-x^2/2}$ and defined inductively by $f_{n+1}(x) = f_n'(x) - xf_n(x)$. Let's change things just a little, just to get rid of pesky minus signs, and define:
$$h_0(x) = e^{-x^2/2}$$
and
$$h_{n+1}(x) = xh_n(x) - h_n'(x) \qquad (*)$$
(so h_n will be $(-1)^n f_n$, and this will save us a lot of $(-1)^n$'s along the way).

(a) Show that h_n is an eigenfunction of the Fourier transform with eigenvalue $(-i)^n\sqrt{2\pi}$, in other words
$$\widehat{h_n}(\omega) = (-i)^n\sqrt{2\pi}h_n(\omega).$$

(b) Explain why h_n is equal to a polynomial of degree n times $e^{-x^2/2}$, say $h_n(x) = H_n(x)e^{-x^2/2}$.

(c) Find the next six of these polynomials, after $H_0(x) = 1$.

(d) Show inductively (or at least give a good argument to convince yourself and others) that
$$xh_n + h_n' = 2nh_{n-1}$$
(if we set $h_{-1}(x) = 0$, this works for $n = 0$, too).

(e) Put the definition (*) of h_n together with part (d) and show that
$$h_n'' - x^2h_n + (2n+1)h_n = 0.$$
(This shows that h_n is an eigenfunction of the operator $L[f] = f'' - x^2 f$ with eigenvalue $2n+1$).

(f) Using (e), show that h_n is orthogonal to h_m on the whole line if $n \neq m$ with respect to the standard inner product
$$\langle f\,,\,g\rangle = \int_{-\infty}^{\infty} f(x)g(x)\,dx.$$

The differential equation $y'' - x^2y + \lambda y = 0$ is called Hermite's equation – and its eigenfunctions $h_n(x)$ are called Hermite functions (and the polynomials $H_n(x)$ are called Hermite polynomials. They arise when you use parabolic coordinates ($x = s^2 - t^2$, $y = 2st$) to separate variables in

Laplace's equation, as well as in the study of the quantum harmonic oscillator (which we'll talk about in class if there's time).

(a) From the formula for the Fourier transform of the Gaussian, we already know that

$$\widehat{h_0}(\omega) = \mathcal{F}\left[e^{-x^2/2}\right](\omega) = \sqrt{2\pi}e^{-\omega^2/2} = (-i)^0\sqrt{2\pi}h_0(\omega).$$

Now, suppose we already know for some value of n that

$$\widehat{h_n}(\omega) = (-i)^n\sqrt{2\pi}h_n(\omega).$$

Then

$$\begin{aligned}
\widehat{h_{n+1}}(\omega) &= \mathcal{F}\left[xh_n(x) - h_n'(x)\right](\omega) \\
&= i\frac{d}{d\omega}[\widehat{h_n}](\omega) - i\omega\widehat{h_n}(\omega) \\
&= -i\left((-i)^n\sqrt{2\pi}\omega h_n(\omega) - (-i)^n\sqrt{2\pi}h_n'(\omega)\right) \\
&= (-i)^{n+1}\sqrt{2\pi}(\omega h_n(\omega) - h_n'(\omega)) \\
&= (-i)^{n+1}\sqrt{2\pi}h_{n+1}(\omega).
\end{aligned}$$

We have shown that if the statement of the problem is true for n then it is true for $n+1$, and we know it's true for $n = 0$. Therefore the statement is true for all $n \geq 0$ by mathematical induction.

(b) We can also prove inductively that $h_n(x)$ is a polynomial $H_n(x)$ of degree n times $e^{-x^2/2}$. Since

$$h_0(x) = e^{-x^2/2} = 1 \cdot e^{-x^2/2},$$

we have $H_0(x) = 1$ so the statement is true for $n = 0$.
Now assume the statement is true for some value of n, so

$$h_n(x) = H_n(x)e^{-x^2/2}$$

with $H_n(x)$ a polynomial of degree n. Then

$$\begin{aligned}
h_{n+1}(x) &= xh_n(x) - h_n'(x) \\
&= xH_n(x)e^{-x^2/2} - (H_n(x)e^{-x^2/2})' \\
&= xH_n(x)e^{-x^2/2} - (H_n'e^{-x^2/2} - xH_ne^{-x^2/2}) \\
&= (2xH_n(x) - H_n'(x))e^{-x^2/2} \\
&= H_{n+1}(x)e^{-x^2/2},
\end{aligned}$$

where $H_{n+1}(x) = 2xH_n(x) - H_n'(x)$. And since $H_n(x)$ is a polynomial of degree n, we have $2xH_n(x)$ is a polynomial of degree $n+1$ and $H_n'(x)$ is a polynomial of degree $n-1$, so their difference is a polynomial of degree $n+1$. (We can also see that the leading coefficient, i.e., the coefficient of x^n, in $H_n(x)$ is 2^n.)

(c) Since $H_{n+1}(x) = 2xH_n - H_n'(x)$, we have

$$\begin{aligned}
H_0(x) &= 1 \\
H_1(x) &= 2xH_0(x) - H_0'(x) = 2x(1) - 0 = 2x \\
H_2(x) &= 2xH_1(x) - H_1'(x) = 2x(2x) - 2 = 4x^2 - 2 \\
H_3(x) &= 2xH_2(x) - H_2'(x) = 2x(4x^2 - 2) - (8x) = 8x^3 - 12x \\
H_4(x) &= 2xH_3(x) - H_3'(x) = 2x(8x^3 - 12x) - (24x^2 - 12) = 16x^4 - 48x^2 + 12 \\
H_5(x) &= 2xH_4(x) - H_4'(x) = 2x(16x^4 - 48x^2 + 12) - (64x^3 - 96x) = 32x^5 - 160x^3 + 120x \\
H_6(x) &= 2xH_5(x) - H_5'(x) = 2x(32x^5 - 160x^3 + 120x) - (160x^4 - 480x^2 + 120) \\
&= 64x^6 - 480x^4 + 720x^2 - 120
\end{aligned}$$

(d) We know that $xh_n(x) - h_n'(x) = h_{n+1}(x)$ and that $h_0(x) = e^{-x^2/2}$. We'll begin by proving a "closed form" formula for h_n as follows. If we look at the equation

$$h_n'(x) - xh_n(x) = -h_{n+1}(x)$$

as though it were a first-order linear differential equation for h_n, then the integrating factor would be $e^{-x^2/2}$, so we multiply through by $e^{-x^2/2}$ and get

$$\frac{d}{dx}\left(e^{-x^2/2}h_n(x)\right) = -e^{-x^2/2}h_{n+1}(x),$$

or

$$h_{n+1}(x) = -e^{x^2/2}\frac{d}{dx}\left(e^{-x^2/2}h_n(x)\right).$$

If we write out the first few h_n's this way, we see that

$$\begin{aligned}
h_0(x) &= e^{-x^2/2} = e^{x^2/2}e^{-x^2} \\
h_1(x) &= -e^{x^2/2}\frac{d}{dx}\left(e^{-x^2/2}h_0(x)\right) = -e^{x^2/2}\frac{d}{dx}\left(e^{-x^2}\right) \\
h_2(x) &= -e^{x^2/2}\frac{d}{dx}\left(e^{-x^2/2}h_1(x)\right) = e^{x^2/2}\frac{d^2}{dx^2}\left(e^{-x^2}\right)
\end{aligned}$$

This leads us to conjecture that

$$h_n(x) = (-1)^n e^{x^2/2}\frac{d^n}{dx^n}\left(e^{-x^2}\right).$$

We've already seen that this is true for $n = 0, 1, 2$ and we now suppose it is true for a certain value of n and attempt to prove that it's true for $n + 1$. So suppose that

$$h_n(x) = (-1)^n e^{x^2/2}\frac{d^n}{dx^n}\left(e^{-x^2}\right).$$

We know that

$$\begin{aligned}
h_{n+1}(x) &= xh_n(x) - h_n'(x) \\
&= -(h_n'(x) - xh_n(x)) \\
&= -e^{x^2/2}\left(e^{-x^2/2}h_n'(x) - xe^{-x^2/2}h_n(x)\right) \\
&= -e^{x^2/2}\frac{d}{dx}\left(e^{-x^2/2}h_n(x)\right) \\
&= -e^{x^2/2}\frac{d}{dx}\left(e^{-x^2/2}\left((-1)^n e^{-x^2/2}\frac{d^n}{dx^n}e^{-x^2}\right)\right) \\
&= (-1)^{n+1}e^{x^2/2}\frac{d^{n+1}}{dx^{n+1}}\left(e^{-x^2}\right),
\end{aligned}$$

which is what we were trying to show. The formula

$$h_n(x) = (-1)^n e^{x^2/2}\frac{d^n}{dx^n}\left(e^{-x^2}\right)$$

is called *Rodruigues's formula* for the Hermite functions.
Now that we have a fairly explicit formula for h_n we should be able to prove the new identity, but we'll need one more little factoid: For any function $f(x)$,

$$\frac{d^n}{dx^n}(xf(x)) = n\frac{d^{n-1}}{dx^{n-1}}(f(x)) + x\frac{d^n}{dx^n}(f(x)).$$

This is certainly true if $n = 0$, and if it's true for n then

$$\begin{aligned}
\frac{d^{n+1}}{d_x^{n+1}}(xf(x)) &= \frac{d}{dx}\left(\frac{d^n}{dx^n}(x(f(x))\right) \\
&= \frac{d}{dx}\left(n\frac{d^{n-1}}{dx^{n-1}}(f(x)) + x\frac{d^n}{dx^n}(f(x))\right) \\
&= n\frac{d^n}{dx^n}(f(x)) + \frac{d^n}{dx^n}(f(x)) + x\frac{d^{n+1}}{dx^{n+1}}(f(x)) \\
&= (n+1)\frac{d^n}{dx^n} + x\frac{d^{n+1}}{dx^{n+1}}(f(x)),
\end{aligned}$$

so it's true for $n+1$ and we've proved the factoid by induction.
Next, use the factoid to note that

$$\begin{aligned}
\frac{d^{n+1}}{dx^{n+1}}e^{-x^2} &= \frac{d^n}{dx^n}\left(\frac{d}{dx}(e^{-x^2})\right) \\
&= \frac{d^n}{dx^n}(-2xe^{-x^2}) \\
&= -2\left(n\frac{d^{n-1}}{dx^{n-1}}e^{-x^2} + x\frac{d^n}{dx^n}e^{-x^2}\right)
\end{aligned}$$

Rearranging this, we get:

$$2x\frac{d^n}{dx^n}e^{-x^2/2} + \frac{d^{n+1}}{dx^{n+1}}e^{-x^2/2} = -2n\frac{d^{n-1}}{dx^{n-1}}e^{-x^2/2}.$$

Break up the first term into two pieces and multiply both sides by $(-1)^n e^{x^2/2}$ and get

$$(-1)^n x e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2} + \left((-1)^n x e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2} + (-1)^n e^{x^2/2} \frac{d^{n+1}}{dx^{n+1}} e^{-x^2/2} \right)$$
$$= (-1)^{n+1} 2n e^{x^2/2} \frac{d^{n-1}}{dx^{n-1}} e^{-x^2/2}.$$

Last, note that $(-1)^{n+1} = (-1)^{n-1}$ and recognize h_n, h_n' and h_{n-1} to conclude

$$x h_n(x) + h_n'(x) = 2n h_{n-1}(x)$$

and we're done. We've certainly learned much more than this last identity along the way!

(e) According to part (d),

$$x h_n(x) + h_n'(x) = 2n h_{n-1}(x).$$

Rewrite this, replacing n by $n+1$, as

$$x h_{n+1}(x) + h_{n+1}'(x) = 2(n+1) h_n(x).$$

Now substitute equation (*)

$$h_{n+1}(x) = x h_n(x) - h_n'(x)$$

for both occurrences of h_{n+1} in the previous equation and get

$$x(x h_n - h_n') + (x h_n - h_n')' = 2(n+1) h_n$$

or

$$(x^2 h_n - x h_n') + (h_n + x h_n' - h_n'') = 2(n+1) h_n$$

which can be algebraically rearranged to

$$h_n'' - x^2 h_n + (2n+1) h_n = 0$$

which is what we are trying to show.

(f) We've done a few of these in the past, the new wrinkle is that the interval of integration is the whole real line from $-\infty$ to ∞. But the procedure is the same. For $n \neq m$, we know that

$$h_n'' - x^2 h_n = -(2n+1) h_n$$

and

$$h_m'' - x^2 h_m = -(2m+1) h_m.$$

We have

$$\begin{aligned}
-(2n+1)\langle h_n, h_m\rangle &= \langle -(2n+1)h_n, h_m\rangle \\
&= \left\langle h_n'' - x^2 h_n, h_m\right\rangle \\
&= \int_{-\infty}^{\infty} (h_n''(x) - x^2 h_n(x))h_m(x)\,dx \\
&= \int_{-\infty}^{\infty} h_n''(x)h_m(x)\,dx - \int_{-\infty}^{\infty} x^2 h_n(x)h_m(x)\,dx \\
&= h_n'(x)h_m(x)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} h_n'(x)h_m'(x)\,dx - \int_{-\infty}^{\infty} x^2 h_n(x)h_m(x)\,dx \\
&= -\,h_n(x)h_m'(x)\Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} h_n(x)h_m''(x)\,dx - \int_{-\infty}^{\infty} x^2 h_n(x)h_m(x)\,dx \\
&= \int_{-\infty}^{\infty} h_n(x)(h_m''(x) - x^2 h_m(x))\,dx \\
&= \left\langle h_n, h_m'' - x^2 h_m\right\rangle \\
&= \langle h_n, -(2m+1)h_m\rangle \\
&= -(2m+1)\langle h_n, h_m\rangle
\end{aligned}$$

where in the middle of this we integrated by parts twice (once with $u = h_m$ and $dv = h_n'\,dx$ and the second time with $u = h_n$ and $dv = h_m'\,dx$) and used the fact that, because h_n and h_m are polynomials times $e^{-x^2/2}$, they and their derivatives approach zero at both ∞ and $-\infty$, so the evaluation terms are zero.

And since we have shown that $(2n+1)\langle h_n, h_m\rangle = (2m+1)\langle h_n, h_m\rangle$ with $n \neq m$, it must be the case that $\langle h_n, h_m\rangle = 0$.