Math 425 / AMCS 525 **Dr. DeTurck**
Assignment 2 **Due Tuesday, January 27, 2016**

Reading: Read the notes on vecctor fields/first-order equations; Sections 1.1, 1.2 of the textbook

Problems: To be handed in at class on Tuesday, January 27.

1. Find the general solution of $u_{xy} = x^2 y$ for the function $u(x, y)$.

2. Ditto for: $yu_{xy} + 2u_x = x$ (Hint: first integrate with respect to x).

3. For the preceding PDE, find the solution that satisfies $u(x, 1) = 0$ and $u(0, y) = 0$.

4. Solve: $u_x + 2u_y = 0$, $5u_x + 6u_y = 0$, $cu_x + du_y = 0$. (These are three separate problems)

5. Solve the equation $yu_x + xu_y = 0$ with $u(0, y) = e^{-y^2}$. In which region of the xy-plane is the solution uniquely determined?

6. Solve $u_x + u_y + u = e^{x+2y}$ with $u(x, 0) = 0$.

7. Solve the equation $u_x + 2u_y + (2x - y)u = 2x^2 + 3xy - 2y^2$.

8. **Oh, Wronski!** Suppose the two functions $y_1(x)$ and $y_2(x)$ are both solutions of the ODE

$$y'' + p(x)y' + q(x)y = 0. \qquad (*)$$

Recall from class that the "Wronskian" of y_1 and y_2 is defined to be the function:

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = y_1(x)y_2'(x) - y_2(x)y_1'(x).$$

(a) Show that W satisfies the first-order equation $W' + pW = 0$.

(b) Solve this equation, and conclude that for any two solutions y_1 and y_2 of (*), either $W(x)$ is zero for all values of x or else $W(x)$ is never zero.

9. Suppose $y(x)$ is a solution of equation (*) from the previous problem. Find a function $v(x)$ so that the function $z(x) = v(x)y(x)$ is a solution of an equation of the form

$$z'' + Q(x)z = 0 \qquad (**)$$

i.e., z satisfies an equation without a first-order term. (Hint: If you do it right, you'll have $Q(x) = q(x) - \frac{1}{4}p(x)^2 - \frac{1}{2}p'(x)$.)

This shows that in order to prove general facts about homogeneous, linear, second-order ODEs, we only need to consider equations of the form (**), since any equation of the form (*) can be converted to (**) by this trick.

10. Now let z be a *non-trivial* (i.e., not identically zero) solution of (**), where $Q(x) < 0$ for all x. Show (by freshman calculus arguments) that there is at most one value of x for which $z(x) = 0$.

11. What happens if $Q(x) > 0$? More on this next week, but you can see that the situation might be complicated from the following example:

Solve the (Cauchy-Euler) equation

$$z'' + \frac{k}{x^2}z = 0$$

and show that every nontrivial solution of has an *infinite* number of zeroes (like sines and cosines do) if $k > 1/4$, but only a finite number (how many are possible?) if $k < 1/4$. What if $k = 1/4$?