Math 425 / AMCS 525 **Dr. DeTurck**
Assignment 2 solutions **Due Tuesday, January 27, 2016**

Reading: Read the notes on vecctor fields/first-order equations; Sections 1.1, 1.2 of the textbook

Problems: To be handed in at class on Tuesday, January 27.

1. Find the general solution of $u_{xy} = x^2 y$ for the function $u(x, y)$.

First integrate with respect to y treating x as a constant, so that the "constant" of integration is allowed to depend on x but not y and get

$$u_x = \frac{x^2y^2}{2} + f(x)$$

Now integrate with respect to x to get:

$$u = \frac{x^3y^2}{6} + \int f(x)\,dx + G(y)$$

Rename $\int f(x)\,dx$ to be $F(x)$, and so the general solution is

$$u(x, y) = \frac{x^3y^2}{6} + F(x) + G(y).$$

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2. Ditto for: $yu_{xy} + 2u_x = x$ (Hint: first integrate with respect to x).

Let $v = u_x$, then the equation becomes $yv_y + 2v = x$. Think of x as a constant, and view this as an ordinary differential equation with y as the independent variable. Write it as

$$v_y + \frac{2}{y}v = \frac{x}{y}$$

so that it is in standard form, and then the solution is

$$\begin{aligned} v &= e^{-\int \frac{2}{y}\,dy} \int \frac{x}{y} e^{\int \frac{2}{y}\,dy}\,dy \\ &= e^{-2\ln y} \int \frac{x}{y} e^{2\ln y}\,dy \\ &= \frac{1}{y^2}\int xy\,dy \\ &= \frac{1}{y^2}\left(\frac{xy^2}{2} + C(x)\right) \\ &= \frac{x}{2} + \frac{C(x)}{y^2} \end{aligned}$$

Since $v = u_x$, now integrate with respect to x and get

$$u = \frac{x^2}{4} + \frac{1}{y^2}\int C(x)\,dx = \frac{x^2}{4} + \frac{1}{y^2}\left(F(x) + g(y)\right)$$

and let $G(y) = g(y)/y^2$ to write the general solution as

$$u(x, y) = \frac{x^2}{4} + \frac{F(x)}{y^2} + G(y).$$

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3. For the preceding PDE, find the solution that satisfies $u(x, 1) = 0$ and $u(0, y) = 0$.

We know that the general solution of the equation in problem 2 is

$$u(x, y) = \frac{x^2}{4} + \frac{F(x)}{y^2} + G(y).$$

Now $u(x, 1) = 0$ means that

$$0 = \frac{x^2}{4} + F(x) + G(1),$$

and $u(0, y) = 0$ means that

$$0 = \frac{F(0)}{y^2} + G(y).$$

The first of these equations tells us that

$$F(x) = -G(1) - \frac{x^2}{4}$$

(so in particular, $F(0) = -G(1)$) and then the second tells us that

$$G(y) = -\frac{F(0)}{y^2} = \frac{G(1)}{y^2}.$$

Put these back into the formula for u and get

$$u(x, y) = \frac{x^2}{4} + \frac{1}{y^2}\left(-G(1) - \frac{x^2}{4}\right) + \frac{G(1)}{y^2} = \frac{x^2}{4} - \frac{x^2}{4y^2}.$$

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4. Solve: $u_x + 2u_y = 0$, $5u_x + 6u_y = 0$, $cu_x + du_y = 0$. (These are three separate problems)

All of these are homogeneous constant-coefficient equations, so the solutions should be constant along the characteristics, which are straight lines. In other words, the general solution is a function of a single variable, which is linear combination of x and y. Suppose $u(x, y) = f(ax + by)$. Then $u_x = af'(ax + by)$ and $u_y = bf'(ax + by)$.

(a) Therefore, for $u_x + 2u_y = 0$, we have $(a + 2b)f'(ax + by) = 0$, so we need $a + 2b = 0$, so $a = 2$ and $b = -1$ will do:

$$u(x, y) = f(2x - y).$$

(b) For $5u_x + 6u_y = 0$, we have $(5a + 6b)f'(ax + by) = 0$, so we need $5a + 6b = 0$ or $a = 6$ and $b = -5$:

$$u(x, y) = f(6x - 5y).$$

(c) For $cu_x + du_y = 0$, we'll need $ca + db = 0$ or $a = d$ and $b = -c$:

$$u(x, y) = f(dx - cy)$$

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5. Solve the equation $yu_x + xu_y = 0$ with $u(0, y) = e^{-y^2}$. In which region of the xy-plane is the solution uniquely determined?

We have to follow the vector field $\mathbf{v} = y\mathbf{i} + x\mathbf{j}$ in the plane, i.e., to solve the system of ordinary differential equations

$$\frac{dx}{dt} = y \quad \frac{dy}{dt} = x \quad \frac{du}{dt} = 0$$

with initial conditions

$$x(0) = 0 \quad y(0) = a \quad u(0) = e^{-a^2}$$

since the initial conditions for the PDE are $u(0, y) = e^{-y^2}$.

Take the derivative of the first ODE and substitute the second to get:

$$\frac{d^2x}{dt^2} = \frac{dy}{dt} = x \qquad \text{and so} \qquad x'' - x = 0.$$

Therefore $x = c_1 \cosh t + c_2 \sinh t$, and $x(0) = 0$ implies $c_1 = 0$ and so $x = c_2 \sinh t$. And from the first equation again we get

$$y = \frac{dx}{dt} = c_2 \cosh t,$$

and $y(0) = a$ implies $c_2 = a$ and therefore

$$x = a \sinh t \quad \text{and} \quad y = a \cosh t$$

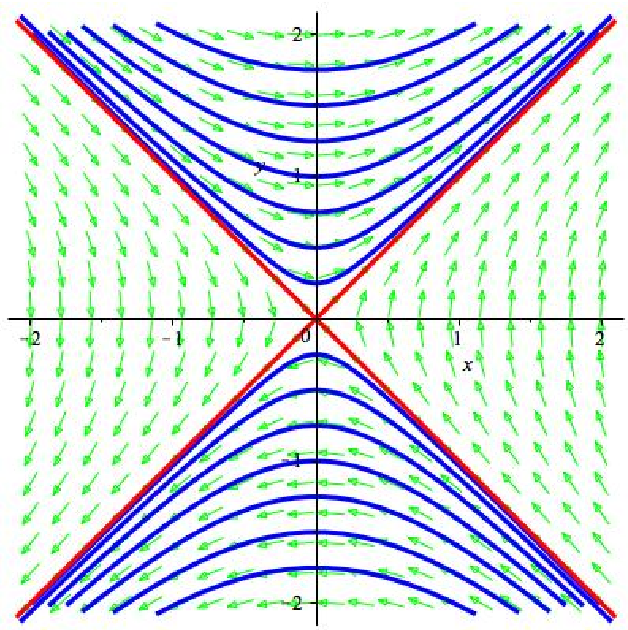
To eliminate a, recall the hyperbolic Pythagorean identity: $\cosh^2 t - \sinh^2 t = 1$, and therefore

$$y^2 - x^2 = a^2(\cosh^2 t - \sinh^2 t) = a^2.$$

Since the solution of the third ODE is $u = e^{-a^2}$ (a constant), we conclude

$$u(x, y) = e^{-(y^2 - x^2)} = e^{x^2 - y^2}.$$

The solution is determined uniquely only at points (x, y) that are "accessible" starting from the y-axis and going along the vector field $\mathbf{v}$. These points are either above $y = |x|$ or below $y = -|x|$ (the red lines in the figure). The solution is not determined for $y \leq |x|$.



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6. Solve $u_x + u_y + u = e^{x+2y}$ with $u(x,0) = 0$.

We'll solve this one twice.

First solution: For the first, we'll solve the system of ODEs:

$$\frac{dx}{dt} = 1 \quad \frac{dy}{dt} = 1 \quad \frac{du}{dt} + u = e^{x+2y}$$

since the first two equations imply

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} = u_x + u_y$$

together with the initial conditions

$$x(0) = c \quad y(0) = 0 \quad u(0) = 0$$

which come from the initial condition $u(x,0) = 0$.

The solutions of the first two ODEs and their initial conditions are easy: $x = t + c$ and $y = t$. Therefore we have to solve

$$u' + u = e^{3t+c} \quad u(0) = 0$$

This is a linear equation already in standard form, so we get the general solution from the formula:

$$u(t) = e^{-\int 1\,dt}\int e^{3t+c}e^{\int 1\,dt}\,dt = e^{-t}\int e^{4t+c}\,dt = e^{-t}\left(\frac{1}{4}e^{4t+c} + K\right) = \frac{1}{4}e^{3t+c} + Ke^{-t}$$

For $u(0) = 0$, we need $K = -\frac{1}{4}e^c$, so the solution is

$$u = \frac{1}{4}e^c(e^{3t} - e^{-t}).$$

To get u in terms of x and y, we recall that $x = t + c$ and $y = t$, so $c = x - y$ and $t = y$. Therefore

$$u(x, y) = \frac{1}{4}e^{x-y}(e^{3y} - e^{-y}) = \frac{1}{4}(e^{x+2y} - e^{x-2y}).$$

Second solution: This solution uses the "coordinate method": The characteristics are lines with slope 1 (they go along the vector field $\mathbf{i} + \mathbf{j}$), i.e., the lines $y = x + a$, so one coordinate should be $a = y - x$. The level curves of the other coordinate should be lines with slope -1, i.e., the lines $y = -x + b$, so the other coordinate is $b = y + x$. In these new coordinates $a = y - x$, $b = y + x$ the equation should simplify. It will help to know how to invert the coordinate transformation: $x = \frac{1}{2}(b - a)$, $y = \frac{1}{2}(b + a)$.

We also have

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial a}\frac{\partial a}{\partial x} + \frac{\partial u}{\partial b}\frac{\partial b}{\partial x} \qquad \text{so} \qquad u_x = -u_a + u_b$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial a}\frac{\partial a}{\partial y} + \frac{\partial u}{\partial b}\frac{\partial b}{\partial y} \qquad \text{so} \qquad u_y = u_a + u_b$$

We see that $u_x + u_y + u = 2u_b + u$ and $x + 2y = \frac{1}{2}(3b + a)$, so the differential equation becomes

$$2u_b + u = e^{\frac{1}{2}(3b+a)}$$

which is essentially an ordinary differential equation with independent variable b. It's linear, and its standard form is

$$u' + \frac{1}{2}u = \frac{1}{2}e^{\frac{1}{2}(3b+a)}.$$

Using the formula for linear ODEs, we get the solution:

$$\begin{aligned}
u &= e^{-\int \frac{1}{2}\,db} \int \frac{1}{2}e^{\frac{1}{2}(3b+a)} e^{\int \frac{1}{2}\,db}\,db \\
&= e^{-\frac{1}{2}b} \int \frac{1}{2}e^{\frac{1}{2}(3b+a)} e^{\frac{1}{2}b}\,db \\
&= e^{-\frac{1}{2}b} \int \frac{1}{2}e^{2b+\frac{1}{2}a}\,db \\
&= e^{-\frac{1}{2}b} \left(\frac{1}{4}e^{2b+\frac{1}{2}a} + C(a)\right) \\
&= \frac{1}{4}e^{\frac{1}{2}(3b+a)} + C(a)e^{-\frac{1}{2}b}
\end{aligned}$$

Now we need to use the initial condition, $u(x, 0) = 0$. This says $0 = \frac{1}{4}e^x + C(-x)e^{-\frac{1}{2}x}$. Solve this for $C(-x)$ and get

$$C(-x) = -\frac{\frac{1}{4}e^x}{e^{-\frac{1}{2}x}} = -\frac{1}{4}e^{\frac{3}{2}x}$$

and replace $-x$ by x (or x by $-x$) to get

$$C(x) = -\frac{1}{4}e^{-\frac{3}{2}x}.$$

Therefore the solution of the initial-value problem is

$$u(x,y) = \frac{1}{4}e^{2y+x} - \frac{1}{4}e^{-\frac{3}{2}(y-x)}e^{-\frac{1}{2}(y+x)} = \frac{1}{4}e^{x+2y} - \frac{1}{4}e^{x-2y}$$

which is what we got before.

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7. Solve the equation $u_x + 2u_y + (2x - y)u = 2x^2 + 3xy - 2y^2$.

First, we'll notice that $2x^2 + 3xy - 2y^2 = (2x - y)(x + 2y)$ (what a coincidence!). Then we'll do the problem two ways.

First solution: For the first, we'll solve the system of ODEs:

$$\frac{dx}{dt} = 1 \quad \frac{dy}{dt} = 2 \quad \frac{du}{dt} + (2x - y)u = (2x - y)(x + 2y)$$

since the first two equations imply

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} = u_x + 2u_y$$

together with the initial conditions

$$x(0) = c \quad y(0) = 0 \quad u(0) = f$$

which come from the more-or-less general iinitial condition $u(x, 0) = f(x)$ (so f is $f(c)$).

The solutions of the first two ODEs and their initial conditions are easy: $x = t + c$ and $y = 2t$, so that $2x - y = 2c$ and $x + 2y = 5t + c$. Therefore we have to solve

$$u' + 2cu = 2c(5t + c) \quad u(0) = f$$

This is a linear equation already in standard form, so we get the general solution from the formula:

$$u(t) = e^{-\int 2c\,dt}\int 2c(5t + c)e^{\int 2c\,dt}\,dt = e^{-2ct}\int 2c(5t + c)e^{2ct}\,dt$$

and we do this last integral by parts: $f = 2c(5t + c)$ and $dg = e^{2ct}\,dt$ so that $df = 10c\,dt$ and $g = \frac{1}{2c}e^{2ct}$, which gives us

$$\begin{aligned} u(t) &= e^{-2ct}\left(2c(5t + c)\frac{1}{2c}e^{2ct} - \int 5e^{2ct}\,dt\right) \\ &= 5t + c - e^{-2ct}\left(\frac{5}{2c}e^{2ct} + K(c)\right) \\ &= 5t + c - \frac{5}{2c} - K(c)e^{-2ct} \end{aligned}$$

Now we need $u = f$ when $t = 0$, which means that

$$f = c - \frac{5}{2c} - K(c) \quad \text{and so} \quad K(c) = f(c) - c + \frac{5}{2c}.$$

In terms of c and t, then, we have

$$u = 5t + c - \frac{5}{2c} - \left(f(c) - c + \frac{5}{2c}\right) e^{-2ct}.$$

Now use that $x = t + c$ and $y = 2t$, or in other words $t = \frac{1}{2}y$ and $c = x - \frac{1}{2}y$ to go back to x and y:

$$u = \frac{5}{2}y + x - \frac{1}{2}y - \frac{5}{2x - y} + \left[\text{a function of } c = x - \tfrac{1}{2}y\right] e^{-\frac{1}{2}y(2x-y)}$$

Now, a function of $x - \frac{1}{2}y$ is also a function of $2x - y$, so we will write our general solution as

$$u(x, y) = 2y + x - \frac{5}{2x - y} + F(2x - y)e^{-\frac{1}{2}y(2x-y)}$$

Second solution: This solution uses the "coordinate method": The characteristics are lines with slope 2 (they go along the vector field $\mathbf{i} + 2\mathbf{j}$), i.e., the lines $y = 2x - b$, so one coordinate should be $b = 2x - y$. The level curves of the other coordinate should be lines with slope $-\frac{1}{2}$, i.e., the lines $2y = -x + a$, so the other coordinate is $a = 2y + x$. In these new coordinates $a = 2y + x$, $b = 2x - y$ the equation should simplify. We don't need the formulas for x and y in terms of a and b because of the fortuitous factorization of the right side of the equation: $(2x - y)(x + 2y) = ba$.

Now

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial a}\frac{\partial a}{\partial x} + \frac{\partial u}{\partial b}\frac{\partial b}{\partial x} \qquad \text{so} \qquad u_x = u_a + 2u_b$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial a}\frac{\partial a}{\partial y} + \frac{\partial u}{\partial b}\frac{\partial b}{\partial y} \qquad \text{so} \qquad u_y = 2u_a - u_b$$

We see that $u_x + 2u_y = 5u_a$ so the differential equation becomes

$$5u_a + bu = ba$$

which is essentially a linear ordinary differential equation with independent variable a and its standard form is

$$u_a + \frac{b}{5}u = \frac{ba}{5}.$$

The formula gives:

$$u = e^{-\int \frac{b}{5}\,da} \int \frac{ba}{5} e^{\int \frac{b}{5}\,da}\,da = e^{-\frac{1}{5}ba} \int \frac{ba}{5} e^{\frac{1}{5}ba}\,da$$

and we do this last integral by parts: $f = \frac{1}{5}ba$ and $dg = e^{\frac{1}{5}ba}\,da$ so $df = \frac{1}{5}b\,da$ and $g = \dfrac{5}{b}e^{\frac{1}{5}}ba$, which gives us

$$u = e^{-\frac{1}{5}ba}\left(ae^{\frac{1}{5}ba} - \int e^{\frac{1}{5}ba}\,da\right) = e^{-\frac{1}{5}ba}\left(ae^{\frac{1}{5}ba} - \frac{5}{b}e^{\frac{1}{5}ba} + C(b)\right) = a - \frac{5}{b} + C(b)e^{-\frac{1}{5}ba}$$

Finally, use $a = 2y + x$ and $b = 2x - y$ to write:

$$u(x, y) = x + 2y - \frac{5}{2x - y} + C(2x - y)e^{-\frac{1}{5}(2x^2+3xy-2y^2)}$$

Reconciliation: If you look at the two solutions we got, they do seem a little bit different. The first three terms are the same, but the last term is $F(2x - y)e^{-\frac{1}{2}y(2x-y)}$ in the first solution and

$C(2x-y)e^{-\frac{1}{5}(2x^2+3xy-2y^2)}$ in the second. But by uniqueness, we know they have to be the same. What's with that?

Well, the answer ought to be some relationship between the functions $F(2x-y)$ and $C(2x-y)$, which will make the last terms of the two solutions equal. To find it, set those terms equal, so

$$F(2x-y)e^{-\frac{1}{2}y(2x-y)} = C(2x-y)e^{-\frac{1}{5}(x+2y)(2x-y)}.$$

Solve this for $F(2x-y)$ and get

$$\begin{aligned} F(2x-y) &= C(2x-y)e^{-\frac{1}{5}(x+2y)(2x-y)}e^{\frac{1}{2}y(2x-y)} \\ &= C(2x-y)e^{(2x-y)(-\frac{1}{5}x-\frac{2}{5}y+\frac{1}{2}y)} \\ &= C(2x-y)e^{(2x-y)(\frac{1}{10}y-\frac{1}{5}x)} \\ &= C(2x-y)e^{-\frac{1}{10}(2x-y)^2} \end{aligned}$$

which is in fact a function of $(2x-y)$. So the solutions become the same if we set $F(z) = e^{-\frac{1}{10}z^2}C(z)$.

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8. **Oh, Wronski!** Suppose the two functions $y_1(x)$ and $y_2(x)$ are both solutions of the ODE

$$y'' + p(x)y' + q(x)y = 0. \qquad (*)$$

Recall from class that the "Wronskian" of y_1 and y_2 is defined to be the function:

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = y_1(x)y_2'(x) - y_2(x)y_1'(x).$$

(a) Show that W satisfies the first-order equation $W' + pW = 0$.

(b) Solve this equation, and conclude that for any two solutions y_1 and y_2 of (*), either $W(x)$ is zero for all values of x or else $W(x)$ is never zero.

(a) Start by calculating, and at a strategic moment use that because they solve the equation (*), we have

$$y_1'' = -p(x)y_1' - q(x)y_1 \qquad \text{and} \qquad y_2'' = -p(x)y_2' - q(x)y_2.$$

Here we go:

$$\begin{aligned} W' &= \frac{d}{dx}(y_1(x)y_2' - y_2(x)y_1'(x)) \\ &= y_1(x)y_2''(x) + y_1'(x)y_2'(x) - (y_2(x)y_1''(x) + y_2'(x)y_1'(x)) \\ &= y_1(-py_2' - qy_2) - y_2(-py_1' - qy_1) \\ &= -py_1y_2' + py_2y_2' = -pW \end{aligned}$$

which is just what we wanted.

(b) $W' + p(x)W = 0$ is a linear differential equation in standard form, which we solve:

$$W = e^{-\int p(x)\,dx}\int 0 \cdot e^{\int p(x)\,dx}\,dx = Ce^{-\int p(x)\,dx}$$

for some constant C. Since exponentials are never zero, this formula shows that either W is zero for all x (if $C = 0$) or else $W(x) \neq 0$ for all x (if $C \neq 0$).

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9. Suppose $y(x)$ is a solution of equation (*) from the previous problem. Find a function $v(x)$ so that the function $z(x) = v(x)y(x)$ is a solution of an equation of the form

$$z'' + Q(x)z = 0 \tag{**}$$

i.e., z satisfies an equation without a first-order term. (Hint: If you do it right, you'll have $Q(x) = q(x) - \frac{1}{4}p(x)^2 - \frac{1}{2}p'(x)$.)

This shows that in order to prove general facts about homogeneous, linear, second-order ODEs, we only need to consider equations of the form (**), since any equation of the form (*) can be converted to (**) by this trick.

Start by putting $z = vy$ in the equation $z'' + Qz = 0$. Since $z' = v'y + vy'$ and $z'' = v''y + 2v'y' + vy''$, this gives:

$$v''y + 2v'y' + vy'' + Qvy = 0, \quad \text{or} \quad vy'' + 2v'y' + (v'' + Qv)y = 0$$

Since y'' has a coefficient of 1 in equation (*), divide this last equation by v to get

$$y'' + \frac{2v'}{v}\,y' + \frac{v'' + Qv}{v}\,y = 0,$$

so we're going to need to choose v so that

$$\frac{2v'}{v} = p, \quad \text{or} \quad \ln v = \int \frac{1}{2}p\,dx \quad \text{or} \quad v = e^{\frac{1}{2}\int p\,dx}.$$

That tells us what v is. Next, we have to choose Q so that

$$\frac{v'' + Qv}{v} = q \quad \text{or} \quad Q = \frac{vq - v''}{v}$$

To simplify this somewhat, use that $v' = \frac{1}{2}pv$, so that

$$v'' = \tfrac{1}{2}(pv' + p'v) = \tfrac{1}{2}(p(\tfrac{1}{2}pv) + p'v) = \tfrac{1}{4}p^2v + \tfrac{1}{2}p'v.$$

Finally, insert this into the expression for Q above to get

$$Q = \frac{qv - \frac{1}{4}p^2v - \frac{1}{2}p'v}{v} = q - \tfrac{1}{4}p^2 - \tfrac{1}{2}p'$$

as promised.

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10. Now let z be a *non-trivial* (i.e., not identically zero) solution of (**), where $Q(x) < 0$ for all x. Show (by freshman calculus arguments) that there is at most one value of x for which $z(x) = 0$.

Suppose there are two values of x, call them $x_1 < x_2$, where z is zero, so $z(x_1) = 0$ and $z(x_2) = 0$. We can assume that $z'(x_1) \neq 0$ and $z'(x_2) \neq 0$ because if one of them was zero then z would satisfy an initial-value problem with zero initial data and so would be identically zero by uniqueness. If these derivatives are nonzero, then there must be points between x_1 and x_2 where z is not zero.

That means that either the maximum value of z on the interval $[x_1, x_2]$ is positive, or the minimum value of z on the interval is negative (or both). We show that in fact neither one can happen.

First, suppose that the maximum value of z is positive, and that it occurs at $x = x_0$. At such a maximum we must have $z'(x_0) = 0$ and $z''(x_0) \leq 0$. But from the differential equation, $z''(x_0) = -Q(x_0)z(x_0) > 0$, since Q is negative for all x and $z(x_0) > 0$. This contradiction shows that a maximum point where z is positive such as x_0 cannot exist.

Likewise, suppose that the minimum value of z is negative, and that it occurs at $x = x_0$. At such a minimum we must have $z'(x_0) = 0$ and $z''(x_0) \geq 0$. But from the differential equation, $z''(x_0) = -Q(x_0)z(x_0) < 0$ since both $Q(x_0)$ and $z(x_0)$ are negative. This contradiction shows that a minimum point where z is negative such as x_0 cannot happen.

If z can't have a positive maximum or a negative minimum on the interval $[x_1, x_2]$, then z must be identically zero on that interval (and thus identically zero everywhere, by uniqueness for the initial-value problem). Therefore, if z is not identically zero then there can be at most one value of x where $z(x) = 0$.

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11. What happens if $Q(x) > 0$? More on this next week, but you can see that the situation might be complicated from the following example:

Solve the (Cauchy-Euler) equation

$$z'' + \frac{k}{x^2} z = 0$$

and show that every nontrivial solution of has an *infinite* number of zeroes (like sines and cosines do) if $k > 1/4$, but only a finite number (how many are possible?) if $k < 1/4$. What if $k = 1/4$?

One way to solve a Cauchy-Euler equation is to make a substitution for the independent variable x, and let $x = e^t$ (or $t = \ln x$). Then

$$\frac{dz}{dx} = \frac{dz}{dt}\frac{dt}{dx} = \frac{1}{x}\frac{dz}{dt}$$

and

$$\begin{aligned}
\frac{d^2z}{dx^2} &= \frac{d}{dx}\left(\frac{dz}{dx}\right) \\
&= \frac{1}{x}\frac{d}{dt}\left(\frac{dz}{dx}\right) = \frac{1}{x}\frac{d}{dt}\left(\frac{1}{x}\frac{dz}{dt}\right) \\
&= \frac{1}{x}\left(\frac{1}{x}\frac{d^2z}{dt^2} - \frac{1}{x}\frac{dz}{dt}\right) \qquad \text{because } \frac{dx}{dt} = e^t = x \\
&= \frac{1}{x^2}\left(\frac{d^2z}{dt^2} - \frac{dz}{dt}\right)
\end{aligned}$$

Therefore

$$\frac{d^2z}{dx^2} + \frac{k}{x^2}z = \frac{1}{x^2}\left(\frac{d^2z}{dt^2} - \frac{dz}{dt}\right) + \frac{k}{x^2}z = \frac{1}{x^2}\left(\frac{d^2z}{dt^2} - \frac{dz}{dt} + kz\right)$$

This means that $z(t)$ satisfies the constant-coefficient equation:

$$\frac{d^2z}{dt^2} - \frac{dz}{dt} + kz = 0$$

and to find $z(t)$ we need to find the roots of the polynomial $r^2 - r + k$, which are

$$r = \frac{1 \pm \sqrt{1-4k}}{2}.$$

If $k > \frac{1}{4}$ then the roots are complex, with real part $\frac{1}{2}$ and imaginary part $\frac{1}{2}\sqrt{4k-1}$, so

$$\begin{aligned} z &= e^{\frac{1}{2}t}\Big(c_1 \cos(\tfrac{1}{2}\sqrt{4k-1}\, t) + c_2 \sin(\tfrac{1}{2}\sqrt{4k-1}\, t)\Big) \\ &= \sqrt{x}\Big(c_1 \cos(\tfrac{1}{2}\sqrt{4k-1}\, \ln x) + c_2 \sin(\tfrac{1}{2}\sqrt{4k-1}\, \ln x)\Big) \end{aligned}$$

and so z has infinitely many zeros, which occur at all the values of x where

$$\tan(\tfrac{1}{2}\sqrt{4k-1}\, \ln x) = -\frac{c_1}{c_2}.$$

If $k < \frac{1}{4}$ then the roots are real and different – call them r_1 and r_2 and the solution in terms of t will be

$$z = c_1 e^{r_1 t} + c_2 e^{r_2 t}.$$

It's enough to show that this expression can't be zero twice, since then the version in terms of x won't be able to either. So suppose there are two values of t, say t_1 and t_2, such that $z(t_1) = z(t_2) = 0$. Then we can view

$$\begin{aligned} c_1 e^{r_1 t_1} + c_2 e^{r_2 t_1} &= 0 \\ c_1 e^{r_1 t_2} + c_2 e^{r_2 t_2} &= 0 \end{aligned}$$

as a system of two linear equations in the two unknowns c_1 and c_2. The determinant of the coefficient matrix is $e^{r_1 t_1 + r_2 t_2} - e^{r_2 t_1 + r_1 t_2}$ which would have to be zero in order for there to be a non-zero solution. But if the determinant is zero then $r_1 t_1 + r_2 t_2 = r_2 t_1 + r_1 t_2$, or in other words $(r_1 - r_2)t_1 = (r_1 - r_2)t_2$. But since r_1 and r_2 are different, this would imply that $t_1 = t_2$, so the two values of t where $z = 0$ are the same, and so z can have at most one zero.

Finally, if $k = \frac{1}{4}$ then the roots are real and equal, $r = \frac{1}{2}, \frac{1}{2}$. In this case the solution in terms of t is

$$z = c_1 e^{\frac{1}{2}t} + c_2 t e^{\frac{1}{2}t}.$$

It is also the case that this expression can't be zero twice, since if $z(t_1) = z(t_2) = 0$ we could view

$$\begin{aligned} c_1 e^{\frac{1}{2}t_1} + c_2 t_1 e^{\frac{1}{2}t_1} &= 0 \\ c_1 e^{\frac{1}{2}t_2} + c_2 t_2 e^{\frac{1}{2}t_2} &= 0 \end{aligned}$$

as a system of two linear equations in the two unknowns c_1 and c_2. This time the determinant of the coefficient matrix is

$$e^{\frac{1}{2}t_1} t_2 e^{\frac{1}{2}t_2} - e^{\frac{1}{2}t_2} t_1 e^{\frac{1}{2}t_1} = e^{\frac{1}{2}(t_1+t_2)}(t_2 - t_1).$$

If the determinant is zero then we'd have $t_2 = t_1$ again, so there can't be two zeros unless the solution is identically zero ($c_1 = c_2 = 0$).

The conclusion of all this is that the nontrivial solutions of the equation $z'' + \dfrac{k}{x^2}z = 0$ have infinitely many zeros of $k > \frac{1}{4}$ but at most one zero if $k \le \frac{1}{4}$.