Math 425 / AMCS 525 **Dr. DeTurck**
Assignment 3 solutions **Due Tuesday, February 2, 2016**

Reading: Read sections 1.3, 1.4, 1.5, 2.1 and 2.2 of the text

Problems: Be prepared to discuss the following problems in class:

1. Page 19, problem 2

You can follow the derivation of the wave equation on pp. 11–12 of the text, except in the case of the hanging chain of length ℓ, the tension T at the point x (length units) below the ceiling is $\rho(\ell - x)g$, where ρ is the (linear) density of the chain (the mass of the part of the chain below x is $\rho(\ell - x)$ so its weight is $\rho(\ell - x)g$). So the horizontal (transverse) force on the part of the chain between $x = x_0$ and $x = x_1$ is

$$\int_{x_0}^{x_1} \rho u_{tt}\,dx = \left.\frac{Tu_x}{\sqrt{1+u_x^2}}\right|_{x_0}^{x^1} = \left.\frac{\rho(\ell - x)gu_x}{\sqrt{1+u_x^2}}\right|_{x_0}^{x_1}$$

Assuming u and u_x are small enough so that $\sqrt{1+u_x^2} \approx 1$ we can differentiate the preceding at x_1 to get

$$[\rho(\ell - x)gu_x]_x = \rho u_{tt} \qquad \text{or} \qquad u_{tt} = g[(\ell - x)u_x]_x.$$

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2. Page 19, problem 4

Let $u(x, y, z, t)$ be the density (concentration) of the particles, then the mass of the particles in some region R of the medium is

$$m_R(t) = \iiint\limits_R u\,dx\,dy\,dz$$

and the rate of change of the mass in the region is

$$\frac{dm_R(t)}{dt} = \iiint\limits_R u_t\,dx\,dy\,dz = \iint\limits_{\mathrm{bd}(R)} \mathbf{F}\cdot(-\mathbf{n})\,d\sigma$$

where $\mathrm{bd}(R)$ is the boundary of the region R, the outward-pointing unit normal vector on $\mathrm{bd}(R)$ is $\mathbf{n}$, and the flux of the particles is given by $\mathbf{F}(x, y, z, t)$. Part of the flux is due to diffusion (so proportional and in the opposite direction to ∇u) and the other part due to the constant-velocity motion of the particles, so $-Vu\mathbf{k}$, where V is the magnitude of the velocity (flux being velocity times density). So we have

$$\frac{dm_R(t)}{dr} = \iint\limits_{\mathrm{bd}(R)} \mathbf{F}\cdot\mathbf{n}\,d\sigma = \iint\limits_{\mathrm{bd}(R)} (-k\nabla u - Vu\mathbf{k})\cdot(-\mathbf{n})\,d\sigma = \iiint\limits_R \nabla\cdot(k\nabla u + Vu\mathbf{k})\,dx\,dy\,dz.$$

Since R is arbitrary, we can conclude that

$$u_t = \nabla\cdot(k\nabla u + Vu\mathbf{k}) = k\Delta u + Vu_z.$$

But since the problem assumes homogeneity in the x and y directions, we have $u_{xx} = u_{yy} = 0$ and so

$$u_t = ku_{zz} + Vu_z.$$

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3. Page 19, problem 5

This is like problem 4. If $u(x,t)$ is the density (concentration) of the dye in the medium, then the mass of dye in an interval $[x_0, x_1]$ at time t is given by

$$m_{x_0,x_1}(t) = \int_{x_0}^{x_1} u(x,t)\,dx$$

and the rate of change of mass in the region is

$$\frac{dm_{x_0,x_1}(t)}{dt} = \int_{x_0}^{x_1} u_t\,dx = -\mathbf{F}(x_1,t) + \mathbf{F}(x_0,t)$$

where $\mathbf{F}$ is the flux of the particles (from left to right). Part of the flux is due to diffusion (so proportional to $-u_x$ by Fick's law) and the other part due to the constant-velocity motion of the whole thing, so Vu since the velocity is to the right. Differentiating the equation with respect to x_1 we get

$$u_t = ku_{xx} - Vu_x.$$

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4. Page 19, problem 6

If we start from $u_t = k\Delta_u$ and use that u depends only on t and r, then we can use the expression for Δu we found on the first homework to get

$$u_t = k\left(u_{rr} + \frac{1}{r}u_r\right)$$

On the other hand, it's instructive to derive that equation from "first principles" as follows: We can ignore the z axis since u doesn't depend on z. Now consider the annulus A between the two circles $r = a$ and $r = b$ in the xy-plane. A measure of the total heat contained in the annulus A is

$$H_A(t) = \iint_A u\,d(\text{area})$$

where $u(x,y,t) = u(r,\theta,t)$ is the temperature. Or in polar coordinates

$$H_A(t) = \iint_A u\,d(\text{area}) = \int_a^b \int_0^{2\pi} u(r,t)\,r\,dr\,d\theta = \int_a^b u(r,t)\,2\pi r\,dr.$$

The rate of change of the heat in A is thus

$$\frac{dH_A(t)}{dt} = \int_a^b 2\pi r u_t\,dr.$$

The rate of change of heat in A is also the flux into/out of the bounding circles. But since u is independent of θ, we have $\nabla u = u_r\mathbf{r}$ where $\mathbf{r}$ is the *unit* vector pointing outward from the origin (as opposed to the vector that points from the origin to the point, which is often denoted $\mathbf{r}$, but which would be $r\mathbf{r}$ the way we've defined $\mathbf{r}$). Since $\mathbf{r}$ is also the outward-pointing normal vector for the outer circle ($r = b$) and the inward-pointing normal for the inner circle $r = a$, and since the flux is in the opposite direction to ∇u, we have that the flux through $r = b$ is $k2\pi b u_r(b)$ and the flux through $r = a$ is $-k2\pi a u_r(a)$. Therefore

$$\frac{dH_A(t)}{dt} = \int_a^b u(r,t)\, 2\pi r\, dr. = 2\pi k b u_r(b) - 2\pi k a u_r(a) = 2\pi k \int_a^b \frac{d}{dr}(r u_r)\, dr = 2\pi k \int_a^b r u_{rr} + u_r\, dr.$$

Since the integrals (of $r u_t$ and $r u_{rr} + u_r$) are equal for all a and b, the integrands are equal, so

$$2\pi r u_{tt} = 2\pi k (r u_{rr} + u_r) \qquad \text{or} \qquad u_{tt} = k\left(u_r r + \frac{1}{r} u_r\right).$$

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5. Page 19, problem 9

This problem is to verify the divergence theorem for the vector field $\mathbf{F} = r^2(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$ on the ball D of radius a centered at the origin. First the left side: Since

$$\nabla \cdot \mathbf{F} = (3x^2 + y^2 + z^2) + (x^2 + 3y^2 + z^2) + (x^2 + y^2 + 3z^2) = 5r^2,$$

we have

$$\iiint_D \nabla \cdot \mathbf{F}\, d(\text{vol}) = \int_0^a \int_0^\pi \int_0^{2\pi} (5r^2) r^2 \sin\varphi\, d\theta\, d\varphi\, dr = 4\pi \int_0^a 5r^4\, dr = 4\pi a^5.$$

On the other hand, since $\mathbf{F} = r^3\mathbf{r}$ (why?), and $\mathbf{r}$ is the outward-pointing normal, we have

$$\iint_{\text{bd}(D)} \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iint_{\text{bd}(D)} a^3 \mathbf{r} \cdot \mathbf{r}\, d\sigma = a^3(\text{area of bd}(D)) = 4\pi a^5$$

since $\text{bd}(D)$ is the sphere of radius a, which has surface area $4\pi a^2$. So the two sides are equal and the divergence theorem is verified in this case.

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6. Page 24, problem 1

Since $u(x, 0) = x^2$, let's try $u(x,t)$ in the form $u(x,t) = p(t) + x^2$. Then $u_t = u_{xx}$ becomes $p'(t) = 2$, and we need $p(0) = 0$ to satisfy the initial condition. So $u(x,t) = 2t + x^2$ satisfies this initial-value problem.

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7. Page 24, problem 3

The steady-state temperature will be a solution of $\Delta u = 0$, with $\partial u/\partial n = 0$. One would expect (correctly) that the steady-state temperature distribution will be constant C at all points (which certainly satisfies the equation and boundary condition, and we'll show later that there is no other solution). What is C? Since heat is neither gained nor lost, the total heat at the end should be the total heat at the beginning, so

$$\iiint_D C\, d(\text{vol}) = \iiint_D f(\mathbf{x})\, d(\text{vol}) = \iiint_D C\, d(\text{vol}) = C\, \text{vol}(D).$$

Therefore

$$C = \frac{1}{\mathrm{vol}(D)} \iiint\limits_D f(\mathbf{x})\, d(\mathrm{vol}).$$

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8. Page 27, problem 1

The general solution of $u'' + u = 0$ is $u = c_1 \sin x + c_2 \cos x$. In order to have $u(0) = 0$ we need $c_2 = 0$, so we are left with the other boundary condition $u(L) = c_1 \sin L = 0$. Certainly $c_1 = 0$ (which gives $u(x) \equiv 0$) is a solution, but if $\sin L = 0$, i.e., if $L = \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$, then c_1 can be anything, so for these values of L the solution is not unique. But if L is not an integer multiple of π, the $u = 0$ is the only solution.

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9. Page 27, problem 2

(a) Start the uniqueness proof the usual way: If u_1 and u_2 are two solutions, then their difference $v = u_1 - u_2$ satisfies the differential equation $v'' + v' = 0$ together with the boundary conditions $v'(0) = v(0) = \frac{1}{2}(v'(\ell) + v(\ell))$. The general solution of the differential equation (from the auxiliary equation $r^2 + r = 0$) is $v = c_1 + c_2 e^{-x}$. For this general solution, we calculate the three quantities that appear in the boundary conditions and learn that we need:

$$-c_2 = c_1 + c_2 = \tfrac{1}{2}(-c_2 e^{-\ell} + c_1 + c_2 e^{-\ell}),$$

or

$$-c_2 = c_1 + c_2 = \tfrac{1}{2} c_1.$$

If we choose any value for c_1, say $c_1 = s$, then $c_2 = -\frac{1}{2}s$ and you can see that all three quantities are equal (to $\frac{1}{2}s$). So the solution is not unique: for any value of s we can add $s - \frac{1}{2}se^{-x}$ to a solution of the original problem and get another, different solution.

(b) Since the quantity $u' + u$ appears in the boundary conditions, and in fact the boundary conditions imply that $u'(\ell) + u(\ell) = u'(0) + u(0)$, and since $u' + u$ is the integral of $u'' + u'$, which appears in the differential equation, we are motivated (yeah, right. . .) to integrate both sides of the differential equation from 0 to ℓ:

$$\int_0^\ell f(x)\, dx = \int_0^\ell u''(x) + u'(x)\, dx = u'(x) + u(x)\Big|_0^\ell = 0$$

since $u'(\ell) + u(\ell) = u'(0) + u(0)$. Therefore, unless

$$\int_0^\ell f(x)\, dx = 0$$

there cannot be a solution.

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10. Page 38, problem 1

To solve $u_{tt} = c^2 u_{xx}$ with $u(x,0) = e^x$ and $u_t(x,0) = \sin x$ we can just use d'Alembert's solution with $\varphi(x) = e^x$ and $\psi(x) = \sin x$:

$$\begin{aligned} u(x,t) &= \frac{1}{2}[\varphi(x+ct) + \varphi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(q)\,dq \\ &= \frac{1}{2}(e^{x+ct} + e^{x-ct}) + \frac{1}{2c}\int_{x-ct}^{x+ct} \sin q\,dq \\ &= e^x \frac{e^{ct} + e^{-ct}}{2} - \frac{1}{2c}[\cos(x+ct) - \cos(x-ct)] \\ &= e^x \cosh ct - \frac{1}{2c}[(\cos x \cos ct - \sin x \sin ct) - (\cos x \cos ct + \sin x \sin ct)] \\ &= e^x \cosh ct + \frac{1}{c} \sin x \sin ct \end{aligned}$$

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11. Page 38, problem 3

We know the speed of the signal will be $c = \sqrt{T/\rho}$, the edge of the signal (where the hammer hits the string) starts at distance $\frac{1}{2}\ell - a$ from the end of the string, and we want to know when the signal reaches position $\frac{1}{4}\ell$. Well, time equals distance over velocity, so

$$t = \frac{(\frac{1}{2}\ell - a) - \frac{1}{4}\ell}{\sqrt{\frac{T}{\rho}}} = \sqrt{\frac{\rho}{T}}\left(\frac{\ell}{4} - a\right)$$

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12. Page 38, problem 9

We can factor the operator as: $u_{xx} - 3u_{xt} - 4u_{tt} = (D_x^2 - 3D_xD_t - 4D_t^2)u = (D_x - 4D_t)(D_x + D_t)u$. So let $v = (D_x + D_t)u = u_x + u_t$, and then v will have to satisfy the PDE

$$(D_x - 4D_t)v = v_x - 4v_t = 0 \quad \text{with the initial condition} \quad v(x,0) = u_x(x,0) + u_t(x,0) = 2x + e^x.$$

To solve this, we need to solve the system of ODEs:

$$\frac{dx}{ds} = 1 \quad \frac{dt}{ds} = -4 \quad \frac{dv}{ds} = 0$$

with initial conditions

$$x(0) = r \quad t(0) = 0 \quad v(0) = 2r + e^r.$$

So $x = r + s$, $t = -4s$ and $v = 2r + e^r$. Since $r = x - s = x + \frac{1}{4}t$, we have

$$v(x,t) = 2(x + \tfrac{1}{4}t) + e^{x + \frac{1}{4}t}.$$

Now we need to solve

$$u_x + u_t = 2(x + \tfrac{1}{4}t) + e^{x+\frac{1}{4}t} \quad \text{with initial condition} \quad u(x,0) = x^2$$

for u. This time, the system of ODEs is

$$\frac{dx}{ds} = 1 \quad \frac{dt}{ds} = 1 \quad \frac{du}{dx} = 2(x + \tfrac{1}{4}t) + e^{x+\frac{1}{4}t}$$

with initial conditions

$$x(0) = r \quad t(0) = 0 \quad u(0) = r^2.$$

So $x = r + s$, $t = s$ and so

$$\frac{du}{ds} = 2(r + s + \tfrac{1}{4}s) + e^{r+s+\frac{1}{4}s} = 2r + \tfrac{5}{2}s + e^{r+\frac{5}{4}s}.$$

Integrating with respect to s we conclude that

$$u(r, s) = 2rs + \tfrac{5}{4}s^2 + \tfrac{4}{5}e^{r+\frac{5}{4}s} + C(r).$$

In order for $u(r, 0) = r^2$ we need $C(r) = r^2 - \frac{4}{5}e^r$. Therefore

$$u(r, s) = 2rs + \tfrac{5}{4}s^2 + \tfrac{4}{5}e^{r+\frac{5}{4}s} + r^2 - \tfrac{4}{5}e^r.$$

since $s = t$ and $r = x - s = x - t$, we conclude that

$$u(x, t) = 2(x - t)t + \tfrac{5}{4}t^2 + \tfrac{4}{5}e^{x-t+\frac{5}{4}t} + (x - t)^2 - \tfrac{4}{5}e^{x-t} = x^2 + \tfrac{1}{4}t^2 + \tfrac{4}{5}e^{x+\frac{1}{4}t} - \tfrac{4}{5}e^{x-t}.$$

Remarkably, this checks and solves the problem.

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13. Page 41, problem 1

Energy conservation tells us that the quantity

$$E(t) = \frac{1}{2}\int_{-\infty}^{\infty} (\rho u_t^2 + T u_x^2)\, dx$$

is a constant. But if $\varphi \equiv \psi \equiv 0$ then $E(0) = 0$, so we have $E(t) = 0$ for all t. Since E is the integral of a sum of squares, i.e., of a non-negative function, the function itself must be zero. So we conclude that $u_t(x, t) \equiv u_x(x, t) \equiv 0$ for all x and t. Now, for any fixed value of x, since $u_t(x, t)$ is zero and $u(x, 0) = \varphi(x) = 0$, we conclude that $u(x, t) = 0$ for all t. And x was arbitrary so we have $u(x, t) = 0$ for all x and t.

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14. Page 41, problem 2

Start from $u(x, t)$ satisfies $u_{tt} = u_{xx}$ (since $c = 1$) and let $e = \frac{1}{2}(u_t^2 + u_x^2)$ and $p = u_t u_x$.

(a) Calculate:

$$\frac{\partial e}{\partial t} = u_t u_{tt} + u_x u_{xt} \quad \text{and} \quad \frac{\partial p}{\partial x} = u_t u_{xx} + u_x u_{tx}$$

but since $u_{tt} = u_{xx}$ from the wave equation, and since $u_{xt} = u_{tx}$ because mixed partials are equal, these two quantities are equal. Likewise, calculate:

$$\frac{\partial e}{\partial x} = u_t u_{tx} + u_x u_{xx} \quad \text{and} \quad \frac{\partial p}{\partial t} = u_t u_{xt} + u_x u_{tt}$$

and the same reasoning applies to show that these are equal.

(b) We can use the results of part (a) to do this easily: To show that e satisfies the wave equation, compute:

$$\frac{\partial^2 e}{\partial t^2} - \frac{\partial^2 e}{\partial x^2} = \frac{\partial}{\partial t}\left(\frac{\partial e}{\partial t}\right) - \frac{\partial}{\partial x}\left(\frac{\partial e}{\partial x}\right) = \frac{\partial}{\partial t}\left(\frac{\partial p}{\partial x}\right) - \frac{\partial}{\partial x}\left(\frac{\partial p}{\partial t}\right) = \frac{\partial^2 p}{\partial t \partial x} - \frac{\partial^2 p}{\partial x \partial t} = 0$$

because the mixed partial derivatives of p are equal. So e satisfies the wave equation (with $c = 1$). Likewise:

$$\frac{\partial^2 p}{\partial t^2} - \frac{\partial^2 p}{\partial x^2} = \frac{\partial}{\partial t}\left(\frac{\partial p}{\partial t}\right) - \frac{\partial}{\partial x}\left(\frac{\partial p}{\partial x}\right) = \frac{\partial}{\partial t}\left(\frac{\partial e}{\partial x}\right) - \frac{\partial}{\partial x}\left(\frac{\partial e}{\partial t}\right) = \frac{\partial^2 e}{\partial t \partial x} - \frac{\partial^2 e}{\partial x \partial t} = 0$$

because the mixed partial derivatives of e are equal. So p also satisfies the wave equation.

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15. Page 41, problem 3

Let $u(x,t)$ be a solution of the wave equation.

(a) If y is a constant, and $v(x,t) = u(x-y,t)$, then

$$\frac{\partial v}{\partial x}(x,t) = \frac{\partial u}{\partial x}(x-y,t) \quad \text{and} \quad \frac{\partial^2 v}{\partial x^2}(x,t) = \frac{\partial^2 u}{\partial x^2}(x-y,t).$$

Likewise

$$\frac{\partial v}{\partial t}(x,t) = \frac{\partial u}{\partial t}(x-y,t) \quad \text{and} \quad \frac{\partial^2 v}{\partial t^2}(x,t) = \frac{\partial^2 u}{\partial t^2}(x-y,t).$$

So

$$v_{tt}(x,t) - c^2 v_{xx}(x,t) = u_{tt}(x-y,t) - c^2 u_{xx}(x-y,t) = 0.$$

So $v(x,t) = u(x-y,t)$ satisfies the wave equation.

(b) In fact, any derivative of any order of a solution is another solution. Let

$$v(x,t) = \frac{\partial^{k+\ell} u}{\partial x^k \partial t^\ell}(x,t).$$

We'll omit the "(x,t)" in what follows since everything is evaluated at (x,t). Then

$$v_{tt} - c^2 v_{xx} = \frac{\partial^{k+\ell+2} u}{\partial x^k \partial t^{\ell+2}} - c^2 \frac{\partial^{k+\ell+2} u}{\partial x^{k+2} \partial t^\ell} = \frac{\partial^{k+\ell}}{\partial x^k \partial t^\ell}\Big(u_{tt} - c^2 u_{xx}\Big) = 0$$

and v satisfies the wave equation.

(c) This time, let $v(x,t) = u(ax,at)$. Then

$$v_x(x,t) = a u_x(ax,at) \qquad \text{and} \qquad v_{xx}(x,t) = a^2 u_{xx}(ax,at)$$

and also

$$v_t(x,t) = a u_t(ax,at) \qquad \text{and} \qquad v_{tt}(x,t) = a^2 u_{tt}(ax,at)$$

So

$$v_{tt}(x,t) - c^2 v_{xx}(x,t) = a^2(u_{xx}(ax,at) - c^2 u_{tt}(ax,at)) = 0$$

and so v satisfies the wave equation.

Problems: Write up solutions of the following to hand in:

1. Page 19, problem 7

Consider the "spherical shell" between the two spheres $r = a$ and $r = b$ (where r is the spherical coordinate $r = \sqrt{x^2 + y^2 + z^2}$). A measure of the total heat contained in the shell is

$$H(t) = \iiint\limits_{\text{shell}} u \, d(\text{vol})$$

where $u(x, y, z, t)$ is the temperature at the point (x, y, z) at time t. Since u is actually only a function of r and t, we write the integral in spherical coordinates r, φ, θ as

$$\iiint\limits_{\text{shell}} u \, d(\text{vol}) = \int_a^b \int_0^{2\pi} \int_0^{\pi} u(r) r^2 \sin\varphi \, d\varphi \, d\theta \, dr = 4\pi \int_a^b r^2 u(r) \, dr.$$

The rate of change of the heat in the shell is thus

$$\frac{dH(t)}{dt} = 4\pi \int_a^b r^2 u_t \, dr.$$

The rate of change of heat in the shell is also the flux into/out of its bounding spheres. But since u is independent of φ and θ, we have $\nabla u = u_r \mathbf{r}$ where $\mathbf{r}$ is the *unit* vector pointing outward from the origin (as opposed to the vector that points from the origin to the point, which is often denoted $\mathbf{r}$, but which would be $r\mathbf{r}$ the way we've defined $\mathbf{r}$). Since $\mathbf{r}$ is also the outward-pointing normal vector for the outer sphere ($r = b$) and the inward-pointing normal for the inner sphere $r = a$, and since the flux is in the opposite direction to ∇u, we have that the flux through $r = b$ is $k4\pi b^2 u_r(b)$ and the flux through $r = a$ is $-k4\pi a^2 u_r(a)$. Therefore

$$\frac{dH(t)}{dt} = 4\pi \int_a^b r^2 u(r,t) \, dr. = 4\pi k b^2 u_r(b) - 4\pi k a^2 u_r(a) = 4\pi k \int_a^b \frac{d}{dr}(r^2 u_r) \, dr = 4\pi k \int_a^b r^2 u_{rr} + 2r u_r \, dr.$$

Since the integrals (of $r^2 u_t$ and $r^2 u_{rr} + 2r u_r$) are equal for all a and b, the integrands are equal, so

$$4\pi r^2 u_{tt} = 4\pi k (r^2 u_{rr} + 2r u_r) \qquad \text{or} \qquad u_{tt} = k\left(u_r r + \frac{2}{r} u_r\right).$$

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2. Page 19, problem 10

Let B_R be the solid ball of radius R centered at the origin in $\mathbb{R}^3$ (i.e., the set of points (x, y, z) for which $x^2 + y^2 + z^2 \leq R^2$) and $S_R = \text{bd}(B_R) = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$. Then

$$\begin{aligned}
\left| \iiint\limits_{\text{all of } \mathbb{R}^3} \nabla \cdot \mathbf{f} \, d(\text{vol}) \right| &= \lim_{R\to\infty} \left| \iiint\limits_{B_R} \nabla \cdot \mathbf{f} \, d(\text{vol}) \right| \\
&= \lim_{R\to\infty} \left| \iint\limits_{S_R} \mathbf{f} \cdot \mathbf{n} d\sigma \right| \\
&\leq \lim_{R\to\infty} \frac{1}{R^3 + 1} \iint\limits_{S_R} d\sigma = \lim_{R\to\infty} \frac{4\pi R^2}{R^3 + 1} = 0.
\end{aligned}$$

3. Page 25, problem 6

(a) Start from the heat equation: $u_t = ku_{xx}$ Since things are at equilibrium, we should have $u_t = 0$, so u is a function of x alone which in turn satisfies $u_x x = 0$, so u is a *linear* function of x on each part of the beam, but it could be different on each of the two parts:

$$u = \begin{cases} a_1 + b_1 x & \text{if} \quad 0 \le x \le L_1 \\ a_2 + b_2 x & \text{if} \quad L_1 \le x \le L_1 + L_2 \end{cases}$$

and we need to find a_1, b_1, a_2 and b_2. Since there are four constants to determine, we expect to have four conditions to satisfy:

1.	left endpoint	$u(0) = 0$	$\Longrightarrow$	$a_1 = 0$
2.	right endpoint	$u(L_1 + L_2) = T$	$\Longrightarrow$	$a_2 + b_2(L_1 + L_2) = T$
	At the interface ($x = L_1$):			
3.	u is continuous	$\lim_{x \to L_1^-} u(x) = \lim_{x \to L_1^+} u(x)$	$\Longrightarrow$	$a_1 + b_1 L_1 = a_2 + b_2 L_1$
4.	heat flux (ku_x) is continuous	$\lim_{x \to L_1^-} k_1 u_x(x) = \lim_{x \to L_1^+} k_2 u_x(x)$	$\Longrightarrow$	$k_1 b_1 = k_2 b_2$

So there are four linear equations in the four unknowns a_1, b_1, a_2 and b_2. Tedious, but doable. We already know $a_1 = 0$. From the fourth equation we have $b_1 = \dfrac{k_2}{k_1} b_2$. Put this into the third equation and get $a_2 = L_1 \left(\dfrac{k_2}{k_1} - 1 \right) b_2$. Now put this into the second equation to get

$$\left(L_1 \left(\frac{k_2}{k_1} - 1 \right) + L_1 + L_2 \right) b_2 = T$$

which gives us b_2:

$$b_2 = \frac{T}{\frac{k_2}{k_1} L_1 + L_2} = \frac{k_1 T}{k_2 L_1 + k_1 L_2}.$$

Then the fourth equation gives us b_1:

$$b_1 = \frac{k_2}{k_1} b_2 = \frac{k_2 T}{k_2 L_1 + k_1 L^2}.$$

And then the third equation gives us a_2:

$$a_2 = L_1 \left(\frac{k_2}{k_1} - 1 \right) b_2 = L_1 \frac{(k_2 - k_1) T}{k_2 L_1 + k_1 L_2}.$$

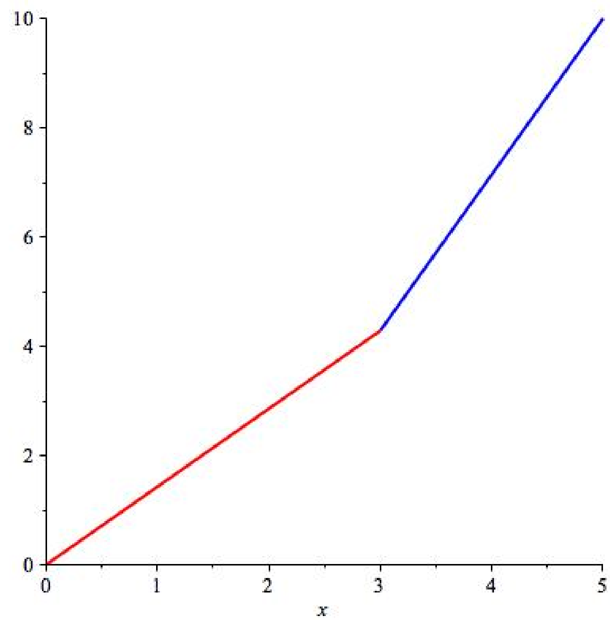
So we conclude that

$$u(x) = \begin{cases} \dfrac{k_2 T}{k_2 L_1 + k_1 L_2} x & \text{if} \quad 0 \le x \le L_1 \\[2ex] L_1 \dfrac{(k_2 - k_1) T}{k_2 L_1 + k_1 L_2} + \dfrac{k_1 T}{k_2 L_1 + k_1 L_2} x & \text{if} \quad L_1 \le x \le L_1 + L_2 \end{cases}.$$

(b) Put $k_1 = 2$, $k_2 = 1$, $L_1 = 3$, $L_2 = 2$ and T=10 into the answer to part (a) and get

$$u(x) = \begin{cases} \dfrac{10}{7}x & \text{if} \quad 0 \le x \le 3 \\ -\dfrac{30}{7} + \dfrac{20}{7}x & \text{if} \quad 3 \le x \le 5 \end{cases}.$$

Here is a graph of the solution:



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4. Page 27, problem 4

(a) Since the differential equation and the boundary conditions involve only derivatives of u (and not u itself), if we add a constant to u neither the Laplacian of u nor the normal derivative of u will change. So if u is a solution, then so is $u + C$ for any constant C.

(b) If $\Delta u = \nabla \cdot \nabla u = f$ and $\dfrac{du}{dn} = \nabla u \cdot \mathbf{n} = 0$ on bd(D), then we will have

$$\iiint_D f\,dx\,dy\,dz = \iiint_D \nabla \cdot \nabla u\,d(\text{vol}) = \iint_{\text{bd}(D)} \nabla u \cdot \mathbf{n}\,d\sigma = \iint_{\text{bd}(D)} \frac{du}{dn}\,d\sigma = 0.$$

(c) Since the normal derivative of u is zero on the boundary of D, this means that no heat (or solute) can flow into or out of the region D, so the total amount of heat (or solute) cannot change. But f represents heat (or solute) sources or sinks — these must sum to zero since the total amount of heat is conserved, since the system is at equilibrium.

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5. Page 38, problem 2

Use d'Alembert's solution

$$u(x,t) = \frac{1}{2}\Big(\varphi(x+ct) + \varphi(x-ct)\Big) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds$$

with $\varphi(x) = \log(1+x^2)$ and $\psi(x) = 4 + x$:

$$\begin{aligned}
u(x,t) &= \frac{1}{2}\log(1+(x+ct)^2) + \frac{1}{2}\log(1+(x-ct)^2) + \frac{1}{2c}\int_{x-ct}^{x+ct} 4 + s\,ds \\
&= \frac{1}{2}\log\Big[(1+(x+ct)^2)(1+(x-ct)^2)\Big] + \frac{1}{2c}\Big[8ct + \frac{1}{2}((x+ct)^2 - (x-ct)^2)\Big] \\
&= \frac{1}{2}\log\Big[(1+(x+ct)^2)(1+(x-ct)^2)\Big] + 4t + xt
\end{aligned}$$

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6. Page 38, problem 5

Use d'Alembert again, with $\varphi(x) = 0$ and $\psi(x) = \begin{cases} 0 & \text{for} \quad x \le -a \\ 1 & \text{for} \quad -a < x < a \\ 0 & \text{for} \quad x > a \end{cases}$. So we have

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s) ds.$$

To evaluate this integral, we make four observations:

- If $x - ct \ge a$, then the interval of integration contains only points where $\psi = 0$, so $u(x,t) = 0$ if $x - ct \ge a$.
- If $x + ct \le -a$, then the same observation applies, so $u(x,t) = 0$ if $x + ct \le a$.
- The length of the interval of integration is $(x+ct) - (x-ct) = 2ct$. If $2ct < 2a$, then there are values of x for which $\psi \equiv 1$ on the interval of integration, so $u(x,t) = \dfrac{2ct}{2c} = t$ here (this is for $ct - a \le x \le a - ct$).
- Likewise, if $2ct > 2a$ them there are values of x where the integral is $2a$ — which is as large as it can be. This happens for $a - ct \le x \le ct - a$, and $u(x,t) = \dfrac{2a}{2c} = \dfrac{a}{c}$ here.

We conclude that:

- For $0 \le t \le \dfrac{a}{c}$,

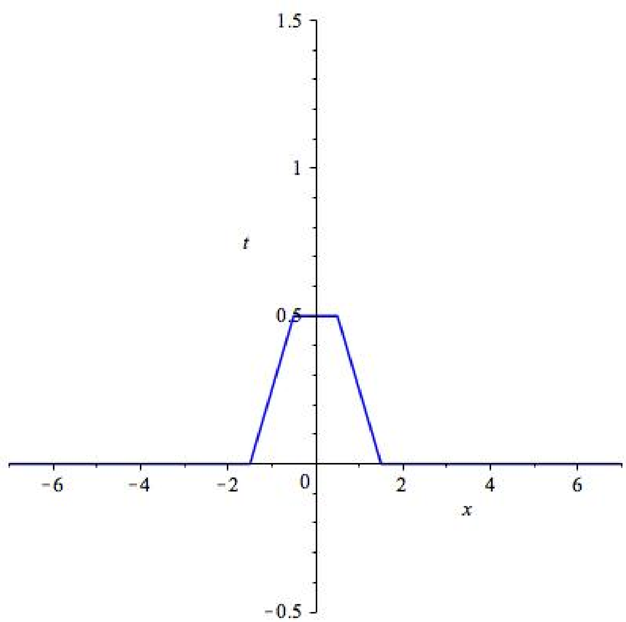
$$u(x,t) = \begin{cases} 0 & \text{if } x < -a - ct \\ \frac{1}{c}(x+a+ct) & \text{if } -a-ct \le x \le -a+ct \\ t & \text{if } -a+ct \le x \le a-ct \\ \frac{1}{c}(a+ct-x) & \text{if } a-ct \le x \le a+ct \\ 0 & \text{if } x \ge a+ct \end{cases}$$

- For $t \geq \dfrac{a}{c}$,

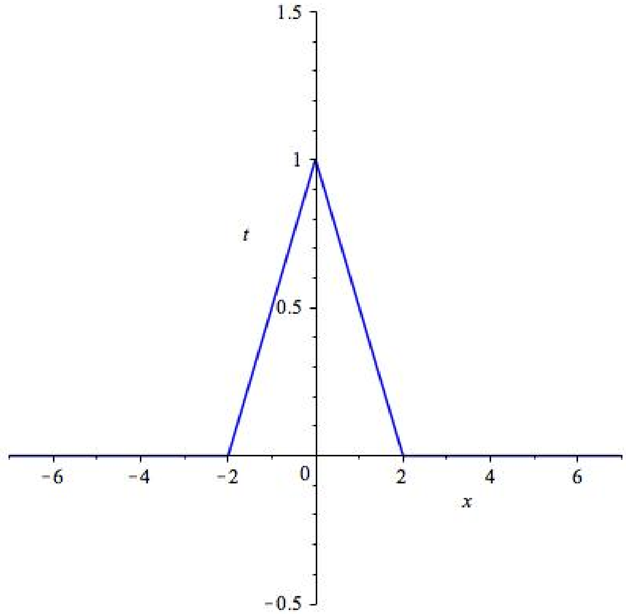
$$u(x,t) = \begin{cases} 0 & \text{if } x < -a - ct \\ \dfrac{1}{c}(x + a + ct) & \text{if } -a - ct \leq x \leq -ct \\ \dfrac{a}{c} & \text{if } -ct \leq x \leq ct \\ \dfrac{1}{c}(a + ct - x) & \text{if } ct \leq x \leq a + ct \\ 0 & \text{if } x \geq a + ct \end{cases}$$

Here are some pictures (with $a = c = 1$):

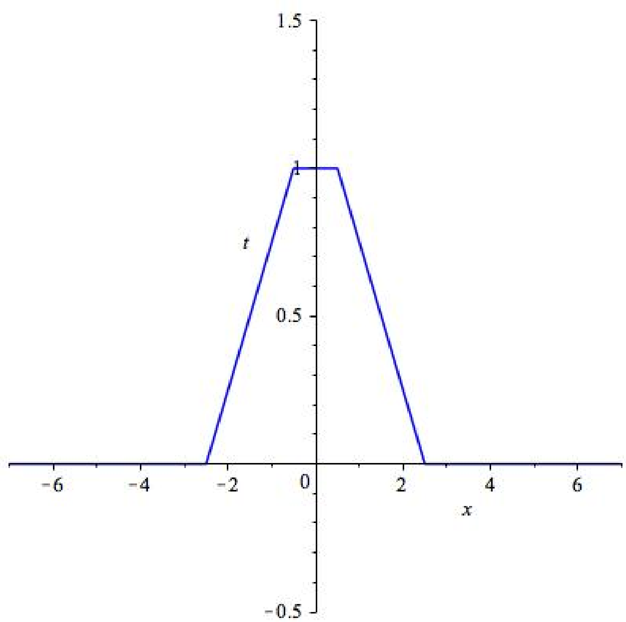
For $t = \dfrac{a}{2c}$ (i.e., $t = \frac{1}{2}$)



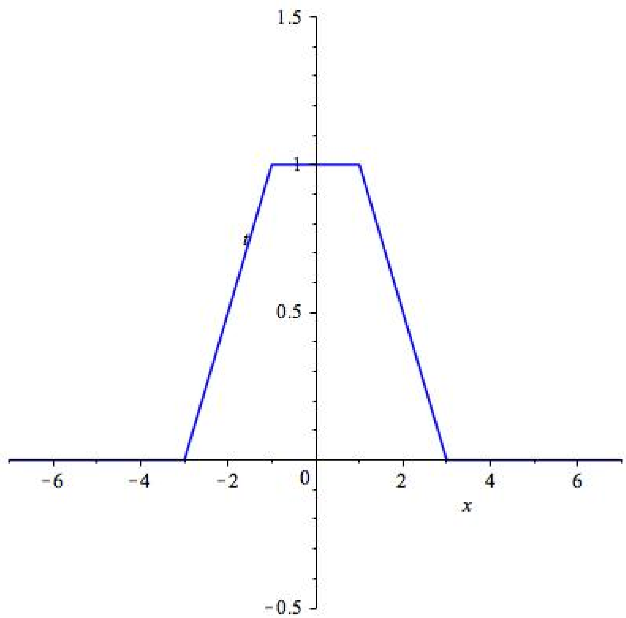
For $t = \dfrac{a}{c}$ (i.e., $t = 1$)



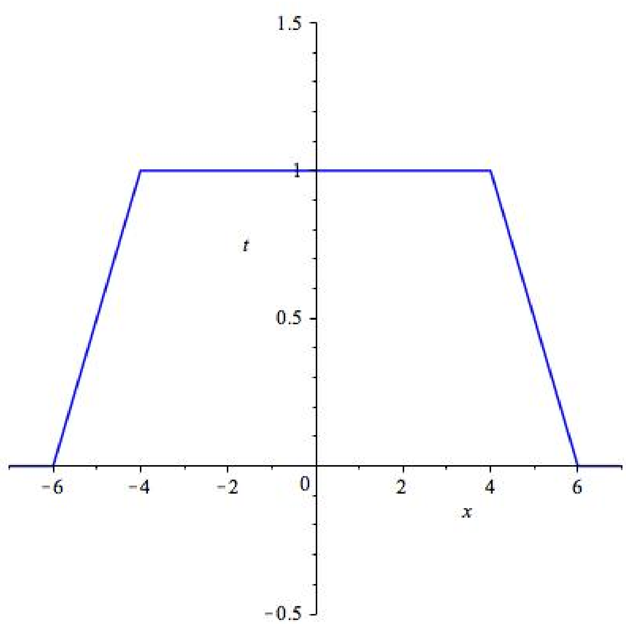
For $t = \dfrac{3a}{2c}$ (i.e., $t = \frac{3}{2}$)



For $t = \dfrac{2a}{c}$ (i.e., $t = 2$)



For $t = \dfrac{5a}{c}$ (i.e., $t = 5$)



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7. Page 38, problem 6

From the analysis in problem 5 above, you can see that the largest value of $u(x, t)$ for any fixed

t is $u(0,t)$, and

$$u(0,t) = \begin{cases} t & \text{if} \quad t < \dfrac{a}{c} \\ \dfrac{a}{c} & \text{if} \quad t \geq \dfrac{a}{c} \end{cases}$$

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8. Page 38, problem 10

You can "factor the operator" as suggested in problem 9, or take a cue from the solution of the wave equation and look for solutions of the form $u(x,t) = f(x + \alpha t)$ for some values of α. Calculate that if u is of this form then

$$u_{xx}(x,t) = f''(x + \alpha t) \qquad u_{xt}(x,t) = \alpha f'(x + \alpha t) \qquad u_{tt}(x,t) = \alpha^2 f''(x + \alpha t).$$

So

$$u_{xx} + u_{xt} - 20u_{tt} = (1 + \alpha - 20\alpha^2) f''(x + \alpha t)$$

For this to be zero with f arbitrary, we need

$$1 + \alpha - 20\alpha^2 = (1 - 4\alpha)(1 + 5\alpha) = 0,$$

so α must be either $\frac{1}{4}$ or $-\frac{1}{5}$. So the general solution of the PDE is

$$u(x,t) = f(x + \tfrac{1}{4}t) + g(x - \tfrac{1}{5}t).$$

Now we need to match the initial conditions:

$$u(x,0) = \varphi(x) = f(x) + g(x)$$
$$u_t(x,0) = \psi(x) = \tfrac{1}{4}f'(x) - \tfrac{1}{5}g'(x)$$

Take the derivative of the first of these equations to get

$$\varphi'(x) = f'(x) + g'(x)$$

and solve the preceding two equations (algebraically — two equations in two unknowns) for $f'(x)$ and $g'(x)$ to get

$$f'(x) = \tfrac{4}{9}\varphi'(x) + \tfrac{20}{9}\psi(x)$$
$$g'(x) = \tfrac{5}{9}\varphi'(x) - \tfrac{20}{9}\psi(x)$$

Now integrate these last two to get

$$f(x) = \tfrac{4}{9}\varphi(x) + \tfrac{20}{9}\int_*^x \psi(s)\,ds$$

$$g(x) = \tfrac{5}{9}\varphi(x) - \tfrac{20}{9}\int_*^x \psi(s)\,ds$$

Using these and the general solution of the PDE above, conclude that:

$$\begin{aligned} u(x,t) &= f(x + \tfrac{1}{4}t) + g(x - \tfrac{1}{5}t) \\ &= \tfrac{4}{9}\varphi(x + \tfrac{1}{4}t) + \tfrac{5}{9}\varphi(x - \tfrac{1}{5}t) + \tfrac{20}{9}\left(\int_*^{x+\frac{1}{4}t} \psi(s)\,ds - \int_*^{x-\frac{1}{5}t} \psi(s)\,ds\right) \\ &= \tfrac{4}{9}\varphi(x + \tfrac{1}{4}t) + \tfrac{5}{9}\varphi(x - \tfrac{1}{5}t) + \tfrac{20}{9}\int_{x-\frac{1}{5}t}^{x+\frac{1}{4}t} \psi(s)\,ds \end{aligned}$$

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9. Page 41, problem 4

Since $u_{tt} = u_{xx}$ we know that

$$u(x,t) = f(x+t) + g(x-t).$$

Therefore:

$$\begin{aligned} u(x+h,t+k) &= f(x+t+h+k) + g(x-t+h-k) \\ u(x-h,t-k) &= f(x+t-h-k) + g(x-t-h+k) \\ u(x+k,t+h) &= f(x+t+h+k) + g(x-t-h+k) \\ u(x-k,t-h) &= f(x+t-h-k) + g(x-t+h-k) \end{aligned}$$

Now the sum of the first two of these and the sum of the last two of these is

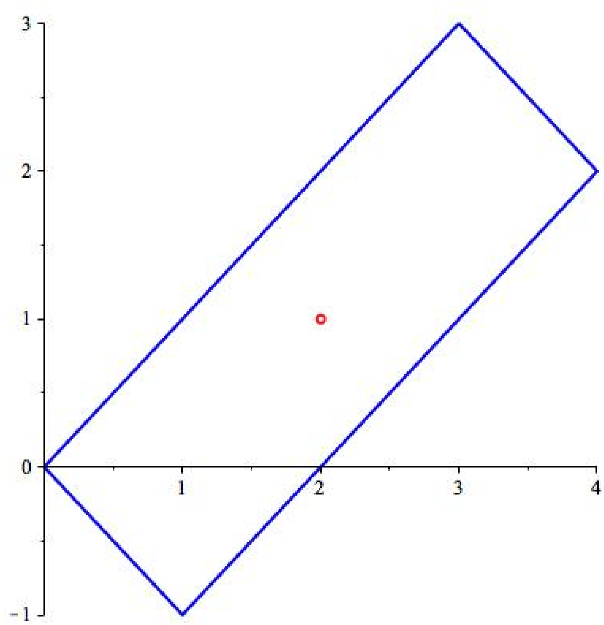
$$f(x+t+h+k) + g(x-t+h-k) + f(x+t-h-k) + g(x-t-h+k)$$

so the sum of the first two is equal to the sum of the last two:

$$u(x+h,t+k) + u(x-h,t-k) = u(x+k,t+h) + u(x-k,t-h)$$

which is what we wanted to show.

Here is a figure with $x = 2$, $t = 1$, $h = 1$ and $k = 2$:



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10. Page 41, problem 5

The equation for the damped string is

$$u_{tt} - c^2 u_{xx} + r u_t = 0$$

where $r > 0$, since the damping provides an acceleration in the opposite direction of the velocity. The energy as a function of time is

$$E(t) = \frac{1}{2} \int_{-\infty}^{\infty} u_t^2 + c^2 u_x^2 \, dx$$

so that

$$\begin{aligned} \frac{dE}{dt} &= \int_{-\infty}^{\infty} u_t u_{tt} + c^2 u_x u_{xt} \, dx \\ &= \int_{-\infty}^{\infty} c^2 u_t u_{xx} - r u_t^2 + c^2 u_x u_{xt} \, dx \end{aligned}$$

using the PDE to replace u_{tt} by $c^2 u_{xx} - r u_t$. Now integrate the last term by parts (assuming the boundary terms vanish at $\pm\infty$) with $f = u_x$ and $dg = u_{xt}$ to get

$$\begin{aligned} \frac{dE}{dt} &= \int_{-\infty}^{\infty} c^2 u_t u_{xx} - r u_t^2 - c^2 u_t u_{xx} \, dx \\ &= -\int_{-\infty}^{\infty} r u_t^2 \, dx \end{aligned}$$

which is non-positive, so the energy decreases (or at least doesn't increase).