Math 425 / AMCS 525 **Dr. DeTurck**
Assignment 4 **Due Tuesday, February 16, 2016**

Topics for this week — The heat equation: initial and initial/boundary value problems for the heat equation, forms of solutions (separated solutions, "fundamental" solution), proofs of uniqueness, maximum principle, continuous dependence.

Fourth Homework Assignment - due Tuesday, February 16

Reading: Read sections 2.3, 2.4,and 3.1 of the text (and any notes that come down the pike)

Be prepared to discuss the following problems in class:

- Page 45 (page 44 in the first ed), problems 2, 4, 6
- Page 52 (page 50 in the first ed), problems 2, 6, 7, 8, 13, 16
- Page 60 (page 58 in the first ed), problems 1, 3

Write up solutions of the following to hand in:

- Page 45 (page 44 in the first ed), problems 2, 4, 6
- Page 52 (page 50 in the first ed), problems 3, 12, 15, 17, 18, 19
- Page 60 (page 58 in the first ed), problems 2, 4

More Wronski!

1. Suppose the two functions $y_1(x)$ and $y_2(x)$ are both solutions of the ODE

$$y'' + p(x)y' + q(x)y = 0. \tag{$*$}$$

Recall from Assignment 2 that the "Wronskian" of y_1 and y_2, which is defined to be the function:

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = y_1(x)y_2'(x) - y_2(x)y_1'(x),$$

is either always zero or never zero. If $W(x) \neq 0$ for all x, then y_1 and y_2 are linearly independent.

Now, assume that y_1 and y_2 are linearly independent solutions of (*), and that $x = a$ and $x = b$ are successive zeroes of y_1, in other words, $y_1(a) = y_1(b) = 0$ and $y_1(x) \neq 0$ for all x such that $a < x < b$. Show that there must be a number c with $a < c < b$ such that $y_2(c) = 0$, in other words, the zeroes of y_1 are separated by the zeroes of y_2 (and vice versa). (Hint: you can assume $y_1(x) > 0$ between a and b [why?]. Use that $W(x) \neq 0$ for all x, which implies that $W(a)$ and $W(b)$ have the same sign, and calculate the signs of $W(a)$ and $W(b)$ from the definition in order to derive a contradiction if y_2 is never zero between a and b.)

2. Let $y(x)$ be a solution of
$$y'' + Q(x)y = 0$$
and let $z(x)$ be a solution of
$$z'' + R(x)z = 0$$
with $R(x) > Q(x) > 0$ for all x. Show that if $x = a$ and $x = b$ are successive zeroes of $y(x)$, then there must be a number c with $a < c < b$ such that $z(c) = 0$ In other words z vanishes at least once between any two successive zeroes of y. (Hint: Use the Wronskian of y and z and an argument similar to the preceding problem. You can argue that $W' > 0$ so W is increasing, even though it has to change from positive to negative, if z doesn't change sign.)

3. Now consider the Bessel equation of index p:
$$x^2y'' + xy' + (x^2 - p^2)y = 0.$$
Call the solution of this $y_p(x)$.

(a) Show that the Bessel equation is equivalent (by the transformation of problem 9 of Assignment 2) to
$$z'' + \left(1 + \frac{1 - 4p^2}{4x^2}\right) z = 0.$$

(b) Use the preceding problem and the equation $y'' + y = 0$ and show that if $0 \leq p < 1/2$, then every interval of length π contains at least one zero of $y_p(x)$, if $p = 1/2$ then the difference between successive zeroes is exactly π, and if $p > 1/2$ then every interval of length π contains at most one zero of $y_p(x)$.

(c) Show that as $x \to \infty$, the distance between successive zeroes of $y_p(x)$ approaches π for every value of p.