Math 425 / AMCS 525 **Dr. DeTurck**
Assignment 4 **Due Tuesday, February 16, 2016**

Topics for this week — The heat equation: initial and initial/boundary value problems for the heat equation, forms of solutions (separated solutions, "fundamental" solution), proofs of uniqueness, maximum principle, continuous dependence.

Fourth Homework Assignment - due Tuesday, February 16

Reading: Read sections 2.3, 2.4,and 3.1 of the text (and any notes that come down the pike)

Be prepared to discuss the following problems in class:

- Page 45 (page 44 in the first ed), problems 2, 4, 6
- Page 52 (page 50 in the first ed), problems 2, 6, 7, 8, 13, 16
- Page 60 (page 58 in the first ed), problems 1, 3

Page 45, problem 2. It actually doesn't matter that u is a solution of the differential equation. If $T_1 > T_2$, the maximum $M(T_1)$ is taken over a bigger set that that for $M(T_2)$, so $M(T_1)$ can't be smaller than $M(T_2)$. Likewise the minimum $m(T_1)$ is taken over a bigger set, so $m(T_1) \le m(T_2)$.

Page 45, problem 4.

(a) By the maximum principle, we know that the maximum and minimum of $u(x,t)$ must occur either for $x = 0$ or $x = 1$ or $t = 0$ (i.e., among the initial or boundary conditions). On $x = 0$ and $x = 1$, we have that $u \equiv 0$, and for $t = 0$ and $0 < x < 1$ we have $0 < 4x(1-x) < 1$, since the maximum occurs at $x = \frac{1}{2}$. Therefore we know that

$$0 \le u(x,t) \le 1$$

for all $0 < x < 1$ and $0 < t < \infty$.

(b) Let $v(x,t) = u(1-x,t)$. Then $v_t(x,t) = u_t(1-x,t)$, $v_t(x,t) = -u_x(1-x,t)$ and $v_{xx} = u_{xx}(1-x,t)$. Therefore $v_t = v_{xx}$ because the same is true for u. Moreover, $v(x,0) = 4(1-x)x = u(x,0)$, $v(0,t) = v(1,t) = 0$. Therefore v satisfies the same initial-value problem as u, and so by uniqueness, $v \equiv u$. Thus $u(1-x,t) = u(x,t)$.

(c) The total energy in the interval at time t is

$$E(t) = \int_0^1 u(x,t)^2\,dx.$$

Take the derivative and use that u satisfies the heat equation $u_t = u_{xx}$:

$$\frac{dE}{dt} = \frac{d}{dt}\int_0^1 u(x,t)^2\,dx = \int_0^1 \frac{\partial}{\partial t}\Big(u(x,t)^2\Big)\,dx = \int_0^1 2u(x,t)u_t(x,t)\,dx = \int_0^1 2u(x,t)u_{xx}(x,t)\,dx.$$

Now do integration by parts with $f = u$ and $dg = u_{xx}\,dx$ and get

$$\frac{dE}{dt} = 2u(1,t)u_x(1,t) - 2u(0,t)u_x(0,t) - \int_0^1 2(u_x(x,t))^2\,dx.$$

The first two terms are zero because of the boundary conditions $u(0,t) = u(1,t) = 0$, and the third term is clearly negative. So $E'(t) < 0$ and $E(t)$ is a decreasing function of t.

Page 45, problem 6. We are given that $u_t = ku_{xx}$ and $v_t = kv_{xx}$ for all $0 < x < \ell$ and $0 < t < \infty$, and that $u(x,0) \le v(x,0)$ for all $0 < x < \ell$, that $u(0,t) \le v(0,t)$ and $u(\ell,t) \le v(\ell,t)$ for all $0 < t < \infty$.

So let $w(x,t) = v(x,t) - u(x,t)$. Then $w_t = kw_{xx}$ (because the diffusion equation is linear), and

$$w(x,0) = v(x,0) - u(x,0) \ge 0 \quad \text{for} \quad 0 < x < \ell$$

and likewise $w(0,t) \ge 0$ and $w(\ell,t) \ge 0$ for $0 < t < \infty$.

But the minimum principle tells us that the minimum value of w is achieved either at time $t = 0$ or at some time on one of the endpoints, i.e., at $(0,t)$ or (ℓ,t). Since $w \ge 0$ at all these points, we conclude that $w(x,t) \ge 0$ everywhere, which means that $v(x,t) - u(x,t) \ge 0$ everywhere, i.e., that

$$u(x,t) \le v(x,t) \quad \text{for all} \quad 0 < x < \ell \text{ and } 0 < t < \infty.$$

Page 52, problem 2. Using the fundamental solution of the heat equation, we get that

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}}\left(\int_{-\infty}^0 3e^{-(x-y)^2/4kt}\,dy + \int_0^\infty e^{-(x-y)^2/4kt}\,dy\right).$$

To $\mathcal{E}\mathrm{rf}(x)$-ify this solution, make the change of variables

$$y \to p = \frac{x-y}{\sqrt{4kt}} \quad \text{so} \quad dp = \frac{-1}{\sqrt{4kt}}\,dy$$

in both integrals, so we get:

$$\begin{aligned}
u(x,t) &= \frac{1}{\sqrt{4\pi kt}}\left(\int_\infty^{x/\sqrt{4kt}} 3e^{-p^2}\,(-\sqrt{4kt}\,dp) + \int_{x/\sqrt{4kt}}^{-\infty} e^{-p^2}\,(-\sqrt{4kt}\,dp)\right)\\
&= \frac{1}{\sqrt{\pi}}\left(3\int_{x/\sqrt{4kt}}^\infty e^{-p^2}\,dp + \int_{-\infty}^{x/\sqrt{4kt}} e^{-p^2}\,dp\right)\\
&= \frac{1}{\sqrt{\pi}}\left(3\Big(\frac{\sqrt{\pi}}{2} - \int_0^{x/\sqrt{4kt}} e^{-p^2}\,dp\Big) + \Big(\frac{\sqrt{\pi}}{2} + \int_0^{x/\sqrt{4kt}} e^{-p^2}\,dp\Big)\right)\\
&= 2 - \mathcal{E}\mathrm{rf}\left(\frac{x}{\sqrt{4kt}}\right)
\end{aligned}$$

We could also approach this as follows: The function $Q(x,t)$ from the text satisfies the heat equation and $Q(x,0) = 1$ if $x > 0$ and $Q(x,0) = 0$ if $x < 0$. So $u(x,t) = 2Q(-x,t) + 1$ will also satisfy the heat equation as well as $u(x,0) = 2Q(-x,0) + 1 = 2 + 1 = 3$ if $x < 0$ and $u(x,0) = 2Q(-x,0) + 1 = 0 + 1 = 1$ if $x > 0$, so this is the solution of the problem. And since $Q(x,t) = \frac{1}{2} + \frac{1}{2}\,\mathcal{E}\mathrm{rf}\left(\frac{x}{\sqrt{4kt}}\right)$, we have

$$u(x,t) = 2Q(-x,t) + 1 = 1 + \mathcal{E}\mathrm{rf}\left(\frac{-x}{\sqrt{4kt}}\right) + 1 = 2 - \mathcal{E}\mathrm{rf}\left(\frac{x}{\sqrt{4kt}}\right)$$

since $\mathcal{E}\mathrm{rf}$ is an odd function.

Page 52, problem 6. Let $I = \int_0^\infty e^{-x^2}\,dx$. Then

$$I^2 = \left(\int_0^\infty e^{-x^2}\,dx\right)\left(\int_0^\infty e^{-y^2}\,dy\right) = \int_0^\infty\int_0^\infty e^{-x^2-y^2}\,dx\,dy = \int_0^{\pi/2}\int_0^\infty e^{-r^2} r\,dr\,d\theta = \frac{\pi}{4}$$

since the domain of integration in the double integral is the entire first quadrant, which is described as $0 < \theta < \frac{\pi}{2}$ and $0 < r < \infty$ in polar coordinates, the element of area is $dx\,dy = r\,dr\,d\theta$ in polar coordinates, and we used the substitution $u = r^2$ to do the integral. Taking the square root gives

$$\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}.$$

Page 52, problem 7. Since e^{-p^2} is an even function, its integral from $-\infty$ to 0 is equal to its integral from 0 to ∞, so

$$\int_{-\infty}^\infty e^{-p^2}\,dp = 2\frac{\sqrt{\pi}}{2} = \sqrt{\pi}.$$

From the definition of $S(x,t)$, we have

$$\int_{-\infty}^\infty S(x,t)\,dx = \frac{1}{2\sqrt{\pi kt}}\int_{-\infty}^\infty e^{-x^2/4kt}\,dx.$$

If we let $p = x/\sqrt{4kt}$, then $dx = \sqrt{4kt}\,dp$ and the integral becomes:

$$\int_{-\infty}^\infty S(x,t)\,dx = \frac{1}{\sqrt{\pi}}\int_{-\infty}^\infty e^{-p^2}\,dp = \frac{1}{\sqrt{\pi}}\sqrt{\pi} = 1$$

independent of t.

Page 52, problem 8. We have

$$S(x,t) = \frac{1}{\sqrt{4\pi kt}}e^{-x^2/4kt}.$$

Therefore

$$\frac{\partial S}{\partial x} = \frac{-2x}{4kt\sqrt{4\pi kt}}e^{-x^2/4kt}.$$

If $\delta > 0$ and $t > 0$, this derivative is clearly negative for all x in the interval $\delta \le x < \infty$, so the maximum value of $\partial S/\partial x$ on this interval (for a fixed value of t) occurs when $x = \delta$, in other words

$$\max_{\delta\le x<\infty} S(x,t) = S(\delta,t) = \frac{1}{\sqrt{4\pi kt}}e^{-\delta^2/4kt}.$$

We can do the limit of this by L'Hôpital's rule: To simplify notation, let $A = -\delta^2/4k$ and $B = \sqrt{4\pi k}$. Then

$$\lim_{t\to 0+} \frac{e^{-\delta^2/4kt}}{\sqrt{4\pi kt}} = \lim_{t\to 0+} \frac{e^{-A/t}}{B\sqrt{t}} = \frac{1}{B} \lim_{t\to 0+} \frac{1/\sqrt{t}}{e^{A/t}}$$

which is a ∞/∞ limit. Use L'Hôpital's rule to transfom this into:

$$\frac{1}{B} \lim_{t\to 0+} \frac{-\frac{1}{2}t^{-3/2}}{-At^{-2}e^{A/t}} = \frac{1}{B} \lim_{t\to 0+} \frac{1}{2A}\sqrt{t}\, e^{-A/t} = 0$$

because both factors with t in them approach zero as $t \to 0+$.

The situation for $-\infty < x < -\delta$ is similar — $S(x,t)$ is increasing on this interval and so the maximum occurs at $x = \delta$. And since δ is squared in the maximum value, the maximum for $x < -\delta$ is the same as for $x > \delta$, so this also approaches zero as $t \to 0+$.

Page 52, problem 13. (a) Recall that $Q(x,t)$ satisfies $Q_t - kQ_{xx} = 0$ and

$$Q(x,0) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases}$$

Let $R(x,t) = Q(\sqrt{a}\,x, at)$. Then

$$R_x(x,t) = \sqrt{a}Q_x(\sqrt{a}\,x, at) \quad R_{xx}(x,t) = aQ_{xx}(\sqrt{a}\,x, at) \qquad R_t(x,t) = aQ_t(\sqrt{a}\,x, t).$$

So $R_t - kR_{xx} = a(Q_t - kQ_xx) = 0$. Moreover,

$$R(x,0) = Q(\sqrt{a}\,x, 0) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases}$$

since $\sqrt{a}\,x > 0$ when $x > 0$. So R satisfies the same initial-value problem as Q, and so $R \equiv Q$, in other words:

$$Q(\sqrt{a}\,x, at) = Q(x,t).$$

(b) If we let $a = 1/4kt$, then we get

$$Q(x,t) = Q\left(\frac{x}{\sqrt{4kt}}, \frac{1}{4k}\right)$$

which shows that Q is really a function of the single variable $p = x/\sqrt{4kt}$, since $1/4k$ is a constant.

Page 52, problem 16. We wish to solve the initial-value problem:

$$u_t - ku_{xx} + bu = 0\,, \qquad u(x,0) = \varphi(x)$$

on the whole line. Using the hint, let $u(x,t) = e^{-bt}v(x,t)$. Then, using the product rule, we have

$$u_t = -be^{-bt}v(x,t) + e^{-bt}v_t(x,t)$$

as well as

$$u_x = e^{-bt}v_x(x,t) \qquad \text{and} \qquad u_{xx} = e^{-bt}v_{xx}(x,t).$$

Therefore

$$u_t - ku_{xx} + bu = -be^{-bt}v + e^{-bt}v_t - ke^{-bt}v_{xx} + be^{-bt}v = e^{-bt}\Big(v_t - kv_{xx}\Big).$$

And we conclude that $u_t - ku_{xx} + bu = 0$ if and only if $v_t - kv_{xx} = 0$, in other words if v satisfies the ordinary diffusion equation. The initial values for v are $v(x,0) = e^0 u(x,0) = \varphi(x)$, and so

$$v(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt} \varphi(y)\, dy$$

which tells us that

$$u(x,t) = \frac{e^{-bt}}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt} \varphi(y)\, dy.$$

Page 60, problem 1. To solve $u_t = ku_{xx}$ for $t > 0$ and $x > 0$ with $u(x,0) = e^{-x}$ and $u(0,t) = 0$, we make the odd extension of e^{-x}, so define

$$\varphi(x) = \begin{cases} e^{-x} & \text{if } x > 0 \\ -e^{x} & \text{if } x < 0 \end{cases}$$

Then the solution of the problem is (the restriction to $x > 0$ of):

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{\frac{-(x-y)^2}{4kt}} \varphi(y)\, dy = \frac{1}{\sqrt{4\pi kt}} \left(\int_{-\infty}^{0} e^{\frac{-(x-y)^2}{4kt}} (-e^{y})\, dy + \int_{0}^{\infty} e^{\frac{-(x-y)^2}{4kt}} e^{-y}\, dy \right)$$

Let $q = -y$ in the first of these integrals and get:

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \left(-\int_{0}^{\infty} e^{\frac{-(x+q)^2}{4kt}} (e^{-q})\, dq + \int_{0}^{\infty} e^{\frac{-(x-y)^2}{4kt}} e^{-y}\, dy \right)$$

Now in the first integral, the overall exponent (after combining the two exponential factors) is

$$-\left(\frac{(x+q)^2}{4kt} + q \right) = -\frac{1}{4kt}\Big(q^2 + 2(x+2kt)q + x^2 \Big) = -\frac{1}{4kt}\Big((q+x+2kt)^2 - 4xkt - 4k^2t^2 \Big)$$

after completing the square. And in the second integral the overall exponent is

$$-\left(\frac{(x-y)^2}{4kt} + y \right) = -\frac{1}{4kt}\Big(y^2 + 2(-x+2kt)y + x^2 \Big) = -\frac{1}{4kt}\Big((y-x+2kt)^2 + 4xkt - 4k^2t^2 \Big)$$

after completing the square.

So now, we'll factor e^{x+kt} out of the first integral in the latest expression for $u(x,t)$ above, and make the change of variables $p = (q+x+2kt)/\sqrt{4kt}$, so that $dp = dq/\sqrt{4kt}$ to convert it (including the factor of $1/\sqrt{4\pi kt}$) to

$$\frac{1}{\sqrt{\pi}} e^{x+kt} \int_{\frac{x+2kt}{\sqrt{4kt}}}^{\infty} e^{-p^2}\, dp = \frac{e^{x+kt}}{2} \left(1 - \mathcal{E}\text{rf}\left(\frac{x+2kt}{\sqrt{4kt}} \right) \right).$$

Next, we'll factor e^{-x+kt} out of the second integral in the expression for $u(x,t)$, and make the change of variable $r = (y-x+2kt)/\sqrt{4kt}$, so that $dr = dy/\sqrt{4kt}$ to convert it to

$$\frac{1}{\sqrt{\pi}} e^{-x+kt} \int_{\frac{-x+2kt}{\sqrt{4kt}}}^{\infty} e^{-r^2}\, dr = \frac{e^{-x+kt}}{2} \left(1 - \mathcal{E}\text{rf}\left(\frac{-x+2kt}{\sqrt{4kt}} \right) \right).$$

We conclude that

$$u(x,t) = -\frac{e^{x+kt}}{2}\left(1 - \mathcal{E}\text{rf}\left(\frac{x+2kt}{\sqrt{4kt}}\right)\right) + \frac{e^{-x+kt}}{2}\left(1 - \mathcal{E}\text{rf}\left(\frac{-x+2kt}{\sqrt{4kt}}\right)\right)$$
$$= -e^{kt}\sinh x + \frac{e^{x+kt}}{2}\mathcal{E}\text{rf}\left(\frac{x+2kt}{\sqrt{4kt}}\right) - \frac{e^{-x+kt}}{2}\mathcal{E}\text{rf}\left(\frac{-x+2kt}{\sqrt{4kt}}\right)$$

Page 60, problem 3. For the Neumann problem on $0 < x < \infty$, we need to extend the initial data to be an even function, so define

$$\varphi_e(x) = \begin{cases} \varphi(x) & \text{if } x > 0 \\ \varphi(-x) & \text{if } x < 0 \end{cases}$$

Then the solution of the problem is (the restriction to $x > 0$ of):

$$w(x,t) = \frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty} e^{\frac{-(x-y)^2}{4kt}}\varphi_e(y)\,dy = \frac{1}{\sqrt{4\pi kt}}\left(\int_{-\infty}^{0} e^{\frac{-(x-y)^2}{4kt}}\varphi(-y)\,dy + \int_{0}^{\infty} e^{\frac{-(x-y)^2}{4kt}}\varphi(y)\,dy\right)$$

Let $q = -y$ in the first of these integrals and get:

$$w(x,t) = \frac{1}{\sqrt{4\pi kt}}\left(\int_{0}^{\infty} e^{\frac{-(x+q)^2}{4kt}}\varphi(q)\,dq + \int_{0}^{\infty} e^{\frac{-(x-y)^2}{4kt}}\varphi(y)\,dy\right)$$
$$= \frac{1}{\sqrt{4\pi kt}}\left(\int_{0}^{\infty}\left[e^{\frac{-(x+y)^2}{4kt}} + e^{\frac{-(x-y)^2}{4kt}}\right]\varphi(y)\,dy\right)$$

after replacing q by y in the first integral, and this agrees with the formula in the text.

Write up solutions of the following to hand in:

- Page 45 (page 44 in the first ed), problems 2, 4, 6
- Page 52 (page 50 in the first ed), problems 3, 12, 15, 17, 18, 19
- Page 60 (page 58 in the first ed), problems 2, 4

Page 52, problem 3. If $u_t = ku_{xx}$ and $u(x,0) = e^{3x}$, then

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty}\exp\left(-\frac{(x-y)^2}{4kt} + 3y\right)\,dy$$
$$= \frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty}\exp\left(-\frac{x^2 - 3xy + y^2 - 12kty}{4kt}\right)\,dy$$
$$= \frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty}\exp\left(= \frac{(y-x-6kt)^2 - 12ktx - 36k^2t^2}{4kt}\right)\,dy$$
$$= \frac{1}{\sqrt{4\pi kt}}e^{3x+9kt}\int_{-\infty}^{\infty} e^{-(y-6kt-x)^2/4kt}\,dy$$

Now make the substitution $p = (y - 6kt - x)/\sqrt{4kt}$, so that $dp = dy/\sqrt{4kt}$. Then we can conclude that

$$u(x,t)\frac{1}{\sqrt{\pi}}e^{3x+9kt}\int_{-\infty}^{\infty} e^{-p^2}\,dp = e^{3x+9kt}$$

(using the result of problem 7).

Page 52, problem 12. (a) Example 1 on page 50 tells us that $Q(x,t) = \frac{1}{2} + \frac{1}{2}\,\mathcal{E}\mathrm{rf}\left(\frac{x}{\sqrt{4kt}}\right)$.

(b) We have

$$\begin{aligned}
\mathcal{E}\mathrm{rf}(x) &= \frac{2}{\sqrt{\pi}}\int_0^x e^{-p^2}\,dp \\
&= \frac{2}{\sqrt{\pi}}\int_0^x 1 - \frac{p^2}{1!} + \frac{p^4}{2!} - \frac{p^6}{3!} + \cdots\,dp \\
&= \frac{2}{\sqrt{\pi}}\left(x - \frac{x^3}{3} + \frac{x^5}{5\cdot 2!} - \frac{x^7}{7\cdot 3!} + \cdots\right) \\
&\approx \frac{2}{\pi}\left(x - \frac{x^3}{3}\right).
\end{aligned}$$

If $|x| < 1$, then the error in this alternating series is less than the first omitted term, namely $\dfrac{|x^5|}{5\sqrt{\pi}}$.

(c) Putting the series from (b) into the formula in (a) for $Q(x,t)$ gives

$$Q(x,t) = \frac{1}{2} + \frac{1}{\sqrt{\pi}}\frac{x}{\sqrt{4kt}} - \frac{1}{3\sqrt{\pi}}\frac{x^3}{(4kt)^{3/2}} + \cdots$$

and if $\dfrac{|x|}{\sqrt{4kt}} < 1$, the error is less than $\dfrac{|x^5|}{10\sqrt{\pi}(4kt)^{5/2}}$ (the first omitted term).

(d) From the error estimate in (c), if $t \gg x$ then the error is *small*.

Page 52, problem 15. Start the usual way: Suppose there are two solutions, u and v. Then we have

$$u_t - ku_{xx} = f(x,t)\,, \quad u_x(0,t) = g(t) \quad u_x(\ell,t) = h(t) \quad u(x,0) = \varphi(x)$$

and

$$v_t - kv_{xx} = f(x,t)\,, \quad v_x(0,t) = g(t) \quad v_x(\ell,t) = h(t) \quad v(x,0) = \varphi(x)$$

Let $w = u - v$. Subtract the corresponding equations above to get:

$$w_t - kw_{xx} = 0\,, \quad w_x(0,t) = 0 \quad w_x(\ell,t) = 0 \quad w(x,0) = 0$$

Now let

$$E(t) = \frac{1}{2}\int_0^\ell (w(x,t))^2\,dx.$$

Then $E(0) = 0$ because $w(x,0) = 0$, and $E(t) \geq 0$ for all $t \geq 0$ (because E is the integral of something squared). But

$$\frac{dE}{dt} = \int_0^\ell ww_t\,dx = \int_0^\ell ww_{xx}\,dx.$$

Integrate by parts with $f = w$ and $dg = w_{xx}\,dx$ (so $df = w_x\,dx$ and $g = w_x$) and get

$$\frac{dE}{dt} = k\Big(w(\ell,t)w_x(\ell,t) - w(0,t)w_x(0,t)\Big) - \int_0^\ell kw_x^2\,dx = -\int_0^\ell kw_x^2\,dx \le 0$$

(because $w_x(0,t) = w_x(\ell,t) = 0$). Therefore $E(t)$ is a non-increasing function, so $E(t) \le E(0) = 0$ for $t > 0$.

Now we have both $E(t) \le 0$ and $E(t) \ge 0$, therefore $E(t) = 0$ for all $t > 0$. Since $E(t)$ is the integral of w^2, we must have $w \equiv 0$, i.e, $u - v \equiv 0$ or $u \equiv v$. So our two solutions were equal after all, proving uniqueness.

Page 52, problem 17. We wish to solve the initial-value problem:

$$u_t - ku_{xx} + bt^2u = 0\,, \qquad u(x,0) = \varphi(x)$$

on the whole line. Using the hint, let $u(x,t) = e^{-bt^3/3}v(x,t)$. Then, using the product rule, we have

$$u_t = -bt^2e^{-bt^3/3}v(x,t) + e^{-bt^3/3}v_t(x,t)$$

as well as

$$u_x = e^{-bt^3/3}v_x(x,t) \qquad \text{and} \qquad u_{xx} = e^{-bt^3/3}v_{xx}(x,t).$$

Therefore

$$u_t - ku_{xx} + bt^2u = -bt^2e^{-bt^3/3}v + e^{-bt^3/3}v_t - ke^{-bt^3/3}v_{xx} + bt^2e^{-bt^3/3}v = e^{-bt^3/3}\Big(v_t - kv_{xx}\Big).$$

And we conclude that $u_t - ku_{xx} + bt^2u = 0$ if and only if $v_t - kv_{xx} = 0$, in other words if v satisfies the ordinary diffusion equation. The initial values for v are $v(x,0) = e^0u(x,0) = \varphi(x)$, and so

$$v(x,t) = \frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty} e^{-(x-y)^2/4kt}\varphi(y)\,dy$$

which tells us that

$$u(x,t) = \frac{e^{-bt^3/3}}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty} e^{-(x-y)^2/4kt}\varphi(y)\,dy.$$

Page 52, problem 18. We want to solve the initial-value problem

$$u_t - ku_{xx} + Vu_x = 0\,, \qquad u(x,0) = \varphi(x)$$

on the whole line. Using the hint (which says to let $y = x - Vt$, which is equivalent to $x = y + Vt$), let $w(y,t) = u(y+Vt,t)$. Then $w_y(y,t) = u_x(y+Vt,t)$ and $w_{yy} = u_{xx}(y+Vt,t)$. The tricky one is w_t, because there are t's in both u "slots" in the definition of w. Therefore

$$w_t(y,t) = u_x(y+Vt,t)x_t + u_t(u+Vt,t) = Vu_x(y+Vt,t) + u_t(y+Vt,t)$$

since $x_t = V$. Therefore

$$w_t - kw_{yy} = u_t(y+Vt,t) + Vu_x(y+Vt,t) - ku_{xx}(y+Vt,t).$$

So if $w(y,t)$ satisfies the ordinary heat equation, $w_t - kw_{yy} = 0$, then $u(x,t)$, where $x = y + Vt$, will satisfy the equation in the problem. Moreover, since $x = y$ when $t = 0$, if w satisfies the heat equation with initial data $w(y,0) = \varphi(y)$, then $u(x,t) = w(x - Vt, t)$ will satisfy the initial-value problem we want. So let

$$w(y,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(y-z)^2/4kt} \varphi(z)\, dz$$

and conclude that

$$u(x,t) = w(x - Vt, t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-Vt-z)^2/4kt} \varphi(z)\, dz$$

Page 52, problem 19. (a) If $S_2(x,y,t) = S(x,t)S(y,t)$, then

$$\frac{\partial S_2(x,y,t)}{\partial x} = \frac{\partial S(x,t)}{\partial x}\, S(y,t) \qquad \text{and} \qquad \frac{\partial^2 S_2(x,y,t)}{\partial x^2} = \frac{\partial^2 S(x,t)}{\partial x^2}\, S(y,t)$$

Likewise,

$$\frac{\partial S_2(x,y,t)}{\partial y} = S(x,t)\, \frac{\partial S(y,t)}{\partial y} \qquad \text{and} \qquad \frac{\partial^2 S_2(x,y,t)}{\partial y^2} = S(x,t)\, \frac{\partial^2 S(y,t)}{\partial y^2}$$

And by the product rule,

$$\frac{\partial S_2(x,y,t)}{\partial t} = \frac{\partial S(x,t)}{\partial t}\, S(y,t) + S(x,t)\, \frac{\partial S(y,t)}{\partial t}.$$

Therefore,

$$\begin{aligned}
&\frac{\partial S_2(x,y,t)}{\partial t} - k\left(\frac{\partial^2 S_2(x,y,t)}{\partial x^2} + \frac{\partial^2 S_2(x,y,t)}{\partial y^2}\right) \\
&\qquad = \frac{\partial S(x,t)}{\partial t}\, S(y,t) + S(x,t)\, \frac{\partial S(y,t)}{\partial t} - k\left(\frac{\partial^2 S(x,t)}{\partial x^2}\, S(y,t) + S(x,t)\, \frac{\partial^2 S(y,t)}{\partial y^2}\right) \\
&\qquad = \left(\frac{\partial S(x,t)}{\partial t} - k\, \frac{\partial^2 S(x,t)}{\partial x^2}\right) S(y,t) + S(x,t)\left(\frac{\partial S(y,t)}{\partial t} - k\, \frac{\partial^2 S(y,t)}{\partial y^2}\right) \\
&\qquad = 0
\end{aligned}$$

because $S(x,t)$ (and $S(y,t)$) satisfies the diffusion equation in one space variable.

(b) It would help if we had a precise definition of "source function". For the purposes of this problem, we'll define the source function as a function $F(x,y,t)$ that

- Satisfies the two-dimensional diffusion equation $F_t = k(F_{xx} + F_{yy})$ for all $(x,y) \in \mathbb{R}^2$ and $t > 0$
- Satisfies $F(x,y,t) > 0$ for all $(x,y) \in \mathbb{R}^2$ and $t > 0$
- Satisfies $\iint_{\mathbb{R}^2} F(x,y,t)\, dx\, dy = 1$ for all $t > 0$.

From part (a), the function S_2 satisfies the two-dimensional diffusion equation. $S_2(x,y,t)$ is positive because $S(x,t)$ and $S(y,t)$ are positive. And it integrates to 1 because both $S(x,t)$ and $S(y,t)$ do so, and the integral of $S_2(x, y, t)$ over the plane is the product of the integrals of $S(x, t)$ and $S(y, t)$ over the lines.

Page 60, problem 2. The first thing we need to do is get rid of the inhomogeneous boundary condition. So let $v(x, t) = u(x, t) - 1$. Since the derivatives of v are the same as those of u, we have that v satisfies the heat equation (for $x > 0$ and $t > 0$) if and only if u does, with initial condition $v(x, 0) = -1$ for $x > 0$ and boundary condition $v(0, t) = 0$ for $t > 0$. Since the boundary condition is a Dirichlet condition, we need the odd extension of the initial data, and we conclude that

$$v(x,t) = \frac{1}{\sqrt{4\pi kt}} \left(\int_{-\infty}^{0} e^{-\frac{(x-y)^2}{4kt}}\, dy - \int_{0}^{\infty} e^{-\frac{(x-y)^2}{4kt}}\, dy \right).$$

Make the standard substitution $p = (x - y)/\sqrt{4kt}$ so that $dp = -dy/\sqrt{4kt}$ and obtain

$$\begin{aligned} v(x,t) &= \frac{1}{\sqrt{\pi}} \left(\int_{x/\sqrt{4kt}}^{\infty} e^{-p^2}\, dp - \int_{-\infty}^{x/\sqrt{4kt}} e^{-p^2}\, dp \right) \\ &= \frac{1}{2} \left[1 - \mathcal{E}\mathrm{rf}\left(\frac{x}{\sqrt{4kt}} \right) - \left(1 + \mathcal{E}\mathrm{rf}\left(\frac{x}{\sqrt{4kt}} \right) \right) \right] = -\mathcal{E}\mathrm{rf}\left(\frac{x}{\sqrt{4kt}} \right) \end{aligned}$$

And so $u(x,t) = 1 + v(x,t) = 1 - \mathcal{E}\mathrm{rf}\left(\dfrac{x}{\sqrt{4kt}} \right)$.

Page 60, problem 4. (a) Since the formula for $v(x, t)$ is the formula (equation (8) on page 49) for the solution of the initial-value problem for the diffusion equation, it is immediate that v satisfies:

$$v_t = kv_{xx} \qquad \text{for } x \in \mathbb{R} \text{ and } t > 0$$

together with the initial condition

$$v(x,0) = f(x) = \begin{cases} x & \text{for } x > 0 \\ x + 1 - e^{2x} & \text{for } x < 0 \end{cases}$$

(b) If $w = v_x - 2v$, then w also satisfies the diffusion equation (since derivatives and linear combinations of solutions of the diffusion equation also satisfy it), together with the initial condition

$$w(x,0) = v_x(x,0) - 2v(x,0) = f'(x) - 2f(x) = \begin{cases} 1 - 2x & \text{for } x > 0 \\ -1 - 2x & \text{for } x < 0 \end{cases}$$

(c) First, note that $f'(x) - 2f(x) = w(x, 0)$. If $x > 0$ then $-x < 0$ and $w(-x, 0) = -1 - 2(-x) = -1 + 2x = -(1 - 2x) = -w(x, 0)$. Likewise, if $x < 0$ then $-x > 0$ and $w(-x, 0) = 1 - 2(-x) = 1 + 2x = -(-1 - 2x) = -w(x, 0)$. So the initial condition $w(x, 0) = f'(x) - 2f(x)$ is an odd function.

(d) Since $w(x, t) + w(-x, t)$ satisfies the diffusion equation and $w(x, 0) + w(-x, 0) = 0$ because $w(x, 0)$ is odd, we have that $w(x, t) + w(-x, t) \equiv 0$ by uniqueness. Therefore $w(x, t)$ is an odd function of x for all t.

(e) Since $w(x, t)$ is odd for all t and $w(x, t)$ is continuous for $t > 0$, it follows that $w(0, t) = 0$ for $t > 0$. By the definition of w from part (b), this means that $v_x(0, t) + 2v(0, t) = 0$ for all $t > 0$. And we already had from part (a) that $v(x,t)$ satisfies the diffusion equation for all $x \in \mathbb{R}$ and $t > 0$ as well as the initial condition $v(x,0) = x$ for $x > 0$. Taken all together, this means that v satisfies problem $(*)$. Therefore, if $u(x,t)$ for $x > 0$ and $t > 0$ is given by the same formula as v, it will also satisfy problem $(*)$.

More Wronski!

1. Suppose the two functions $y_1(x)$ and $y_2(x)$ are both solutions of the ODE

$$y'' + p(x)y' + q(x)y = 0. \tag{$*$}$$

Recall from Assignment 2 that the "Wronskian" of y_1 and y_2, which is defined to be the function:

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = y_1(x)y_2'(x) - y_2(x)y_1'(x),$$

is either always zero or never zero. If $W(x) \neq 0$ for all x, then y_1 and y_2 are linearly independent.

Now, assume that y_1 and y_2 are linearly independent solutions of (*), and that $x = a$ and $x = b$ are successive zeroes of y_1, in other words, $y_1(a) = y_1(b) = 0$ and $y_1(x) \neq 0$ for all x such that $a < x < b$. Show that there must be a number c with $a < c < b$ such that $y_2(c) = 0$, in other words, the zeroes of y_1 are separated by the zeroes of y_2 (and vice versa). (Hint: you can assume $y_1(x) > 0$ between a and b [why?]. Use that $W(x) \neq 0$ for all x, which implies that $W(a)$ and $W(b)$ have the same sign, and calculate the signs of $W(a)$ and $W(b)$ from the definition in order to derive a contradiction if y_2 is never zero between a and b.)

Since y_1 and y_2 are linearly independent, neither one can be identically zero. Moreover, by uniqueness for the initial-value problem, if $y_1(a) = 0$, then $y_1'(a)$ cannot be zero, since if it were, y_1 would have to be identically zero. Therefore, at any value a of x for which $y_1(a) = 0$, it must be the case the y_1 changes sign (in other words, goes from being negative to positive or from positive to negative as x increases past $a = 0$). Of course, the same is true for y_2.

Now, assume that $y_1(a) = y_1(b) = 0$ and $y_1(x) \neq 0$ for all x between a and b. We can assume that $y_1(x) > 0$ between a and b, since if $y_1(x)$ is a solution of the problem with $y_1(a) = y_1(b) = 0$, then so is $-y_1(x)$, because the differential equation is linear and homogeneous. And since $y_1(x) > 0$ between a and b, we know that $y_1'(a) > 0$ (because y_1 had to have gone from negative to positive at $x = a$) and $y_1'(b) < 0$ (because y_1 is going from positive to negative at b). But now let's compute:

$$W(a) = y_1(a)y_2'(a) - y_2(a)y_1'(a) = 0 - y_2(a)y_1'(a) = -y_2(a)y'1(a)$$

and

$$W(b) = y_1(b)y_2'(b) - y_2(b)y_1'(b) = 0 - y_2(b)y_1'(b) = -y_2(b)y'1(b)$$

Since W is never zero, we know that $W(a)$ and $W(b)$ have the same sign (both positive or both negative). But $y_1'(a)$ and $y_2'(b)$ have opposite signs, which implies that in order for both of the above equations to be true, we need $y_2(a)$ and $y_2(b)$ to have opposite signs, which in turn implies that there is a point $x = c$ between a and b where $y_2(c) = 0$.

Applying this reasoning in reverse shows that there must be a zero of y_1 between any two successive zeros of y_2. Therefore the zeros of y_1 and y_2 "interlace" each other. This is sometimes called *Sturm's separation theorem.*

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2. Let $y(x)$ be a solution of

$$y'' + Q(x)y = 0$$

and let $z(x)$ be a solution of

$$z'' + R(x)z = 0$$

with $R(x) > Q(x) > 0$ for all x. Show that if $x = a$ and $x = b$ are successive zeroes of $y(x)$, then there must be a number c with $a < c < b$ such that $z(c) = 0$ In other words z vanishes at least once between any two successive zeroes of y. (Hint: Use the Wronskian of y and z and an argument similar to the preceding problem. You can argue that $W' > 0$ so W is increasing, even though it has to change from positive to negative, if z doesn't change sign.)

If $x = a$ and $x = b$ are successive zeros of $y(x)$, then we can assume $y(x) > 0$ between $x = a$ and $x = b$ as in the preceding problem, so $y'(a) > 0$ and $y'(b) < 0$. Likewise, if $z(x)$ is never zero for $a < x < b$ we can assume $z(x) > 0$ on this interval. Thus

$$W(a) = y(a)z'(a) - z(a)y'(a) = -z(a)y'(a) \leq 0$$

and

$$W(b) = y(b)z'(b) - z(b)y'(b) = -z(b)y'(b) \geq 0$$

Next, note that

$$W'(x) = y(x)z''(x) - z(x)y''(x) = -y(x)R(x)z(x) + z(x)Q(x)y(x) = (Q(x) - R(x))y(x)z(x)$$

which is *negative* for x between a and b since $R(x) > Q(x) > 0$ and both $y(x) > 0$ and $z(x) > 0$ by assumption. Therefore we must have $W(b) < W(a)$, which contradicts that $W(b)$ is non-negative and $W(a)$ is non-positive. Therefore the assumption that z has no zeroes between a and b is invalid. (What we have proved here is usually called *Sturm's comparison theorem.*)

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3. Now consider the Bessel equation of index p:

$$x^2y'' + xy' + (x^2 - p^2)y = 0.$$

Call the solution of this $y_p(x)$.

(a) Show that the Bessel equation is equivalent (by the transformation of problem 9 of Assignment 2) to

$$z'' + \left(1 + \frac{1 - 4p^2}{4x^2}\right) z = 0.$$

(b) Use the preceding problem and the equation $y'' + y = 0$ and show that if $0 \leq p < 1/2$, then every interval of length π contains at least one zero of $y_p(x)$, if $p = 1/2$ then the difference between successive zeroes is exactly π, and if $p > 1/2$ then every interval of length π contains at most one zero of $y_p(x)$.

(c) Show that as $x \to \infty$, the distance between successive zeroes of $y_p(x)$ approaches π for every value of p.

(a) According to Assignment 2, problem 9, we should first divide by x^2 to get

$$y'' + \frac{1}{x}y' + \frac{x^2 - p^2}{x}y = 0$$

and then let

$$z(x) = e^{\frac{1}{2}\int \frac{1}{x}\,dx} y(x) = \sqrt{x}\, y(x) \quad \text{or} \quad y(x) = \frac{z(x)}{\sqrt{x}}.$$

Then

$$y'(x) = \frac{z'(x)}{\sqrt{x}} - \frac{z(x)}{2x^{3/2}}$$

and

$$y''(x) = \frac{z''(x)}{\sqrt{x}} - \frac{z(x)}{x^{3/2}} + \frac{3z(x)}{4x^{5/2}}$$

and so

$$\begin{aligned} x^2y'' + xy' + (x^2 - p^2)y &= \left(x^{3/2}z''(x) - x^{1/2}z'(x) + \frac{3z(x)}{4\sqrt{x}}\right) + \left(x^{1/2}z'(x) - \frac{z(x)}{2\sqrt{x}}\right) + \left(x^{3/2} - \frac{p^2}{x^{1/2}}\right)z(x) \\ &= x^{3/2}z''(x) + \left(x^{3/2} + \frac{1}{4\sqrt{x}} - \frac{p^2}{\sqrt{x}}\right) z(x) \end{aligned}$$

Since this is supposed to be zero, we can divide through by $x^{3/2}$ and get

$$z'' + \left(1 + \frac{1 - 4p^2}{4x^2}\right) z = 0$$

which is what we were after.

(b) Let's start with $p = \frac{1}{2}$, Then $1 - 4p^2 = 0$ and the equation becomes $z'' + z = 0$, the solutions of which are $c_1 \sin x + c_2 \cos x$. If we rewrite this solution by setting $A^2 = c_1^2 + c_2^2$ and $\theta = \arctan(c_2/c_1)$ so that $c_1 = A\cos\theta$ and $c_2 = A\sin\theta$, then we have

$$z = A(\cos\theta \sin x + \sin\theta \cos x) = A\sin(x + \theta)$$

and it is easy to see that for any (nonzero) value of A and any value of θ, this function of x has zeros at $x = -\theta, -\theta \pm \pi, -\theta \pm 2\pi \ldots$, so there are infinitely many zeros, spaced π apart.

Now, if $0 < p < \frac{1}{2}$, let $R(x) = 1 + \dfrac{1 - 4p^2}{4x^2}$ and let $Q(x) = 1$. Then $R(x) > Q(x)$, and so by problem 2 above we know that there is a zero of a solution of $z'' + R(x)z = 0$ between every pair of successive zeroes of a solution of $y'' + Q(x)y = 0$. But the solutions of the latter are $A\sin(x + \theta)$, so given the interval $[a, a + \pi]$, we can choose $\theta = a$ in our solution of $y'' + Q(x)y = 0$ and then problem 2 guarantees a zero of z on the interval.

On the other hand, if $p > \frac{1}{2}$, then we reverse the situation: let $R(x) = 1$ and $Q(x) = 1 + \dfrac{1 - 4p^2}{4x^2}$. Once again, $R(x) > Q(x)$. so there must be a zero of the solution of $z'' + R(x)z = 0$ (i.e., of $z = A\sin(x + \theta)$ between every pair of successive zeros of $y'' + Q(x)y = 0$. But the zeros of z occur π units apart, so on an interval of length π there can be at most one zero of y.

(c) As $x \to \infty$, the function $R(x) = 1 + \dfrac{1 - 4p^2}{4x^2}$ approaches 1, so eventually either

$$1 \le R(x) \le (1 + \varepsilon)^2$$

or else

$$(1 - \varepsilon)^2 \le R(x) \le 1.$$

In the first case, the zeros of $z'' + R(x)z = 0$ occur more frequently than those of $A\sin(x + a)$ but less frequently than those of $A\sin\Big((1+\varepsilon)x + a\Big)$, so they are more than $\pi/(1+\varepsilon)$ apart but within π of each other — so approaching π units apart. And in the second case the zeros occur less frequently than those of $A\sin(x+a)$ but more frequently than those of $A(\sin\Big((1-\varepsilon)x+a\Big)$, so they are more than π apart but within $\pi/(1-\varepsilon)$ of each other — again approaching π units apart.