Math 425 / AMCS 525 **Dr. DeTurck**
Assignment 5 **Due Tuesday, February 23, 2016**

Topics for this week — Duhamel's principle and other techniques for inhomogeneous problems Solutions on finite domains - separated solutions Boundary conditions – eigenvalue (Sturm-Liouville) problems for ODEs

Fifth Homework Assignment - due Tuesday, February 23

Reading: Read sections 3.2, 3.3, 3.4, 4.1 and 4.2 of the text

Be prepared to discuss the following problems in class:

- Page 66 (page 64 in the first ed), problems 1, 3, 9
- Page 70 (page 68 in the first ed), problem 1
- Page 79 (page 76 in the first ed), problems 1, 4, 10
- Page 89 (page 87 in the first ed), problems 2, 3
- Page 92 (page 90 in the first ed), problem 1

Page 66, problem 1. The Neumann problem for the wave equation on $0 < x < \infty$ is $u_tt = c^2 u_{xx}$ for all $x > 0$ and $t > 0$ subject to the initial conditions $u(x, 0) = \varphi(x)$ and $u_t(x, 0) = \psi(x)$ for $x > 0$ and the boundary condition $u_x(0, t) = 0$ for $t > 0$. To solve this, we need the *even* extensions of $\varphi(x)$ and $\psi(x)$:

$$\varphi(x) = \begin{cases} \varphi(x) & \text{if } x > 0 \\ \varphi(-x) & \text{if } x < 0 \end{cases} \qquad \text{and} \qquad \psi(x) = \begin{cases} \psi(x) & \text{if } x > 0 \\ \psi(-x) & \text{if } x < 0 \end{cases}$$

We need to interpret d'Alembert's solution

$$u(x,t) = \frac{1}{2}\Big(\varphi(x+ct) + \varphi(x-ct)\Big) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds$$

in terms of these extended functions. There are two cases: either $x > ct$ or $x < ct$.

In the first case, $x > ct$, we have $x + ct > 0$ and $x - ct > 0$. In this case,

$$u(x,t) = \frac{1}{2}\Big(\varphi(x+ct) + \varphi(x-ct)\Big) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds$$

In the second case, $x < ct$, we have $x + ct > 0$ and $x - ct < 0$ (or $ct - x > 0$). In this case,

$$\begin{aligned} u(x,t) &= \frac{1}{2}\Big(\varphi(x+ct) + \varphi(ct-x)\Big) + \frac{1}{2c}\left(\int_{x-ct}^{0} \psi(-s)\,ds + \int_0^{x+ct} \psi(s)\,ds\right) \\ &= \frac{1}{2}\Big(\varphi(x+ct) + \varphi(ct-x)\Big) + \frac{1}{2c}\left(\int_0^{ct-x} \psi(q)\,dq + \int_0^{x+ct} \psi(s)\,ds\right) \\ &= \frac{1}{2}\Big(\varphi(x+ct) + \varphi(ct-x)\Big) + \frac{1}{2c}\left(2\int_0^{ct-x} \psi(q)\,dq + \int_{ct-x}^{x+ct} \psi(s)\,ds\right) \end{aligned}$$

where we made the substitution $q = -s$ in one of the integrals. We conclude that

$$u(x,t) = \begin{cases} \frac{1}{2}\Big(\varphi(x+ct) + \varphi(x-ct)\Big) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds & \text{if } x \geq ct \\ \frac{1}{2}\Big(\varphi(x+ct) + \varphi(ct-x)\Big) + \frac{1}{2c}\left(2\int_{0}^{ct-x} \psi(q)\,dq + \int_{ct-x}^{x+ct} \psi(s)\,ds\right) & \text{if } x \leq ct \end{cases}$$

Page 66, problem 3. Since $u(x,t) = f(x+ct)$ for $x > 0$ and $t < 0$, we have

$$u(x,0) = f(x) \quad \text{and} \quad u_t(x,0) = cf'(x) \quad (\text{for } x < 0)$$

We also have $u(0,t) = 0$ because the end is "fixed". Because of this, we extend f and cf' as *odd* functions:

$$\varphi(x) = \begin{cases} f(x) & \text{if } x > 0 \\ -f(-x) & \text{if } x < 0 \end{cases} \qquad \text{and} \qquad \psi(x) = \begin{cases} cf'(x) & \text{if } x > 0 \\ -cf'(-x) & \text{if } x < 0 \end{cases}$$

We need to interpret d'Alembert's solution

$$u(x,t) = \frac{1}{2}\Big(\varphi(x+ct) + \varphi(x-ct)\Big) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds$$

in terms of these extended functions. There are two cases: either $x > ct$ or $x < ct$.

In the first case, $x > ct$, we have $x + ct > 0$ and $x - ct > 0$. In this case,

$$\begin{aligned} u(x,t) &= \frac{1}{2}\Big(f(x+ct) + f(x-ct)\Big) + \frac{1}{2c}\int_{x-ct}^{x+ct} cf'(s)\,ds \\ &= \frac{1}{2}\Big(f(x+ct) + f(x-ct)\Big) + \frac{1}{2}\Big(f(x+ct) - f(x-ct)\Big) \\ &= f(x+ct) \end{aligned}$$

In the second case, $x < ct$, we have $x + ct > 0$ and $x - ct < 0$ (or $ct - x > 0$). In this case,

$$\begin{aligned} u(x,t) &= \frac{1}{2}\Big(f(x+ct) - f(ct-x)\Big) + \frac{1}{2c}\left(\int_{x-ct}^{0} -cf'(-s)\,ds + \int_{0}^{x+ct} cf'(s)\,ds\right) \\ &= \frac{1}{2}\Big(f(x+ct) - f(ct-x)\Big) + \frac{1}{2}\left(\int_{ct-x}^{0} f'(q)\,dq + f(x+ct) - f(0)\right) \\ &= \frac{1}{2}\Big(f(x+ct) - f(ct-x)\Big) + \frac{1}{2}\Big(f(0) - f(ct-x) + f(x+ct) - f(0)\Big) \\ &= f(x+ct) - f(ct-x) \end{aligned}$$

where we made the substitution $q = -s$ in one of the integrals. We conclude that

$$u(x,t) = \begin{cases} f(x+ct) & \text{if } x \geq ct \\ f(x+ct) - f(ct-x) & \text{if } x \leq ct \end{cases}$$

Page 66, problem 9. We want to solve $u_{tt} = u_{xx}$ on the x-interval $0 < x < 1$. Since $c = 1$, the characteristics have slope ± 1. And because the boundary conditions $u(0,t) = u(1,t) = 0$ are homogeneous Dirichlet conditions, we need to make the odd extensions φ_{ext} and ψ_{ext} of the initial data $u(x,0) = x^2(1-x)$ and $u_t(x,0) = (1-x)^2$ at every integer (so these extensions will have period 2). In d'Alembert's solution

$$u(x,t) = \frac{1}{2}\Big(\varphi_{ext}(x+ct) + \varphi_{ext}(x-ct)\Big) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{ext}(s)\, ds$$

we can use the oddness of the extensions to simplify the integral of ψ_{ext}, since whenever we integrate over an x-interval of length 2 we will get zero.

(a) To find $u(\frac{2}{3}, 2)$, we need initial data on the x-interval $[\frac{2}{3} - 2, \frac{2}{3} + 2] = [-\frac{4}{3}, \frac{8}{3}]$. Since this interval has length 4, we know that the integral of the odd periodic extension of ψ over it is zero. So we know that

$$u(\tfrac{2}{3}, 2) = \tfrac{1}{2}(\varphi_{ext}(-\tfrac{4}{3}) + \varphi_{ext}(\tfrac{8}{3})) = -\tfrac{1}{2}(\varphi_{ext}(-\tfrac{2}{3}) + \varphi_{ext}(\tfrac{4}{3})) = \tfrac{1}{2}(\varphi(\tfrac{2}{3}) + \varphi(\tfrac{2}{3})) = u(\tfrac{2}{3}, 0) = \tfrac{4}{27}$$

(b) To find $u(\frac{1}{4}, \frac{7}{2})$, we need initial data on the x-interval $[\frac{1}{4} - \frac{7}{2}, \frac{1}{4} + \frac{7}{2}] = [-\frac{13}{4}, \frac{15}{4}]$. Unfortunately, this time the interval has length 7, so we can't conclude that the integral of ψ_{ext} is zero, but we can simplify it a bit:

$$\begin{aligned}\int_{-13/4}^{15/4} \psi_{ext}(s)\, ds = \int_{-5/4}^{7/4} \psi_{ext}(s)\, ds &= \int_{3/4}^{-1/4} \psi_{ext}(s)\, ds = -\int_{-1/4}^{3/4} \psi_{ext}(s)\, ds \\ &= -\int_{1/4}^{3/4} u_t(s,0)\, ds = \frac{(1-x)^3}{3}\bigg|_{1/4}^{3/4} = \frac{1-27}{192} = -\frac{13}{96}\end{aligned}$$

where between the last two integrals we used that ψ_{ext} is odd so that its integral from $-\frac{1}{4}$ to $\frac{1}{4}$ is zero. We conclude that:

$$\begin{aligned}u(\tfrac{1}{4}, \tfrac{7}{2}) &= \tfrac{1}{2}\Big(\varphi_{ext}(-\tfrac{13}{4}) + \varphi_{ext}(\tfrac{15}{4})\Big) + \tfrac{1}{2}(-\tfrac{13}{96}) = \tfrac{1}{2}\Big(\varphi_{ext}(-\tfrac{5}{4}) + \varphi_{ext}(\tfrac{7}{4})\Big) - \tfrac{13}{192} \\ &= \tfrac{1}{2}\Big(\varphi_{ext}(\tfrac{3}{4}) + \varphi_{ext}(-\tfrac{1}{4})\Big) - \tfrac{13}{192} = \tfrac{1}{2}\Big(u(\tfrac{3}{4}, 0) - u(\tfrac{1}{4}, 0)\Big) - \tfrac{13}{192} \\ &= \tfrac{1}{2}\Big(\tfrac{9}{64} - \tfrac{3}{64}\Big) - \tfrac{13}{192} = \tfrac{3}{64} - \tfrac{13}{192} = -\tfrac{4}{192} = -\tfrac{1}{48}.\end{aligned}$$

Page 70, problem 1. We want to solve $u_t - ku_{xx} = f(x,t)$ for $x > 0$ and $t > 0$ with $u(0,t) = 0$ and $u(x,0) = \varphi(x)$. Since we're dealing with a Dirichlet condition at $x = 0$, we make *odd* extensions of $\varphi(x)$ and $f(x,t)$ so we let

$$F(x,t) = \begin{cases} f(x,t) & \text{for } x > 0 \\ -f(-x,t) & \text{for } x < 0 \end{cases}$$

and

$$h(x) \begin{cases} \varphi(x) & \text{for } x > 0 \\ -\varphi(-x) & \text{for } x < 0 \end{cases}$$

Then

$$u(x,t) = \int_0^t \frac{1}{\sqrt{4\pi k(t-s)}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4k(t-s)}} F(y,s)\,dy\,ds + \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} h(y)\,dy$$

Then, writing $S(x,t) = e^{-x^2/4kt}/\sqrt{4\pi kt}$ as usual, we have

$$\begin{aligned}
u(x,t) &= \int_0^t \int_{-\infty}^{\infty} S(x-y,t-s)F(y,s)\,dy\,ds + \int_{-\infty}^{\infty} S(x-y,t)h(y)\,dy \\
&= \int_0^t \left[\int_0^{\infty} S(x-y,t-s)F(y,s)\,dy + \int_{-\infty}^{0} S(x-y,t-s)F(y,s)\,dy\right] ds \\
&\qquad + \int_0^{\infty} S(x-y,t)h(y)\,dy + \int_{-\infty}^{0} S(x-y,t)h(y)\,dy \\
&= \int_0^t \left[\int_0^{\infty} S(x-y,t-s)f(y,s)\,dy + \int_{-\infty}^{0} S(x-y,t-s)(-f(-y,s))\,dy\right] ds \\
&\qquad + \int_0^{\infty} S(x-y,t)\varphi(y)\,dy + \int_{-\infty}^{0} S(x-y,t)(-\varphi(-y))\,dy \\
&= \int_0^t \left[\int_0^{\infty} S(x-y,t-s)f(y,s)\,dy + \int_0^{\infty} S(x+y,t-s)(-f(y,s))\,dy\right] ds \\
&\qquad + \int_0^{\infty} S(x-y,t)\varphi(y)\,dy + \int_0^{\infty} S(x+y,t)(-\varphi(y))\,dy \\
&= \int_0^t \int_0^{\infty} \frac{1}{\sqrt{4\pi k(t-s)}} \left[e^{-\frac{(x-y)^2}{4k(t-s)}} - e^{-\frac{(x+y)^2}{4k(t-s)}}\right] h(y,s)\,dy\,ds + \frac{1}{\sqrt{4\pi kt}} \int_0^{\infty} \left[e^{-\frac{(x-y)^2}{4k(t)}} - e^{-\frac{(x+y)^2}{4k(t)}}\right] \varphi(y)\,dy
\end{aligned}$$

Page 79, problem 1. Since the initial position and velocity are both zero, the solution of $u_{tt} = c^2 u_{xx} + xt$ is given by

$$\begin{aligned}
u(x,t) &= \frac{1}{2c} \int_0^t \int_{x-ct+cs}^{x+ct-cs} ys\,dy\,ds = \frac{1}{2c} \int_0^t \frac{1}{2} y^2 s \Big|_{y=x-ct+cs}^{y=x+ct-cs} ds \\
&= \frac{1}{4c} \int_0^t 4x(ct-cs)s\,ds = \frac{1}{4c} \left[2xcts^2 - \frac{4}{3}xcs^3\right]_{s=0}^{s=t} \\
&= \frac{1}{4c} \left[2xct^3 - \frac{4}{3}xct^3\right] = \frac{1}{6}xt^3
\end{aligned}$$

Page 79, problem 4. The formula for the solution of

$$u_{tt} = c^2 u_{tt} + f(x,t) \quad u(x,0) = 0 \;\text{ and }\; u_t(x,0) = \varphi(x) \text{ for } x > 0, \quad \text{and} \quad u(0,t) = \psi(x)$$

is given in Theorem 1 on page 71 of the text:

$$u(x,t) = \frac{1}{2}[\varphi(x+ct) + \varphi(x-ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(s)\,ds + \frac{1}{2c} \iint_{\Delta} f.$$

And as you can see:

$$u(x,t) = \boxed{\tfrac{1}{2}[\varphi(x+ct)+\varphi(x-ct)]} + \boxed{\frac{1}{2c}\int_{x-ct}^{x+ct}\psi(s)\,ds} + \boxed{\frac{1}{2c}\iint_{\Delta} f}$$

it is the sum of three terms, one involving φ, one involving ψ and one involving f.

Page 79, problem 10. Just for practice, we'll use the Green's function method. So suppose that $u(x,t)$ (for $x > 0$ and $t > 0$) is the solution of

$$u_{tt} = c^2 u_{tt} + f(x,t) \quad u(x,0) = 0 \text{ and } u_t(x,0) = 0 \text{ for } x > 0, \quad \text{and} \quad u(0,t) = 0$$

The domain of dependence D for $u(x_0,t_0)$ is either the triangle with vertices $(x_0-ct_0,0)$, $(x_0+xt_0,0)$, and (x_0,t_0) if $x_0 > ct_0$, or else it is the quadrilateral with vertices $(ct_0-x_0,0)$, $(x_0+ct_0,0)$, (x_0,t_0) and $(0,t_0-x_0/c)$. For $x_0 > ct_0$ we already know (see the notes) that

$$u(x_0,t_0) = \frac{1}{2c}\iint_D f(y,s)\,dy\,ds.$$

For $x_0 < ct_0$, we have to prove this. So here goes: Assume that $u(x,t)$ is the solution of the problem, and that $ct > x$. Then D is the quadrilateral described above and

$$\iint_D f(y,s)\,dy\,ds = \iint_D u_{tt} - c^2 u_{xx} = \oint_{\text{bd}(D)} -u_t\,dx - c^2 u_x\,dt$$

The evaluation of the line integral has four parts:

- The horizontal segment on the x-axis starting at $(ct_0-x_0,0)$ and ending at $(x_0+ct_0,0)$. This part evaluates to zero because $u_t(x,0) = 0$ by the initial conditions and $dt = 0$ on the segment since t is identically zero there.
- The diagonal segment from $(x_0+ct_0,0)$ to (x_0,t_0). We have to do this one. Parametrize the segment via

  $$x = x_0 + c(t_0 - s) \quad \text{and} \quad t = s \quad \text{for} \quad 0 \le s \le t_0$$

  and then $dx = -c\,ds$ and $dt = ds$ so the integral becomes

  $$\int_{(x_0+ct_0,0)}^{(x_0,t_0)} -u_t\,dx - c^2 u_x\,dt = \int_0^{t_0} c u_t(x_0+c(t_0-s),s) - c^2 u_x(x_0+c(t_0-s),s)\,ds$$

  Now we need an observation: with x and t defined as functions of s as in the parametrization, we have that

  $$\frac{du}{ds} = \frac{\partial u}{\partial x}\frac{dx}{ds} + \frac{\partial u}{\partial t}\frac{dt}{ds} = -cu_x(x_0+c(t_0-s),s) + u_t(x_0+c(t_0-s),s).$$

  This is precisely $1/c$ times the integrand, so the value of this part of the line integral is the difference between the value of $cu(x,t)$ at the end of the segment and at the beginning of it. In other words:

  $$\int_{(x_0+ct_0,0)}^{(x_0,t_0)} -u_t\,dx - c^2 u_x\,dt = \frac{1}{c}\Big(u(x_0,t_0) - u(x_0+ct_0,0)\Big) = c\,u(x_0,t_0)$$

  because of the initial condition $u(x,0) = 0$.

- The diagonal segment from (x_0, t_0) to the point $(0, t_0 - x_0/c)$ on the t-axis. We parametrize this segment via
$$x = x_0 - cs \quad \text{and} \quad t = t_0 - s \quad \text{for} \quad 0 \le s \le \frac{x_0}{c}$$
and then $dx = -c\,ds$ and $dt = -ds$ so the integral becomes
$$\int_{(x_0,t_0)}^{(0,t_0-x_0/c)} -u_t\,dx - c^2 u_x\,dt = \int_0^{x_0/c} cu_t(x_0 - cs, t_0 - s) + c^2 u_x(x_0 - cs, t_0 - s)\,ds$$
We make an observation as on the previous segment: with x and t defined as functions of s as in the parametrization of this segment, we have that
$$\frac{du}{ds} = \frac{\partial u}{\partial x}\frac{dx}{ds} + \frac{\partial u}{\partial t}\frac{dt}{ds} = -cu_x(x_0 - cs, t_0 - s) - u_t(x_0 - cs, t_0 - s).$$
And this is precisely $-1/c$ times the integrand, so the value of this part of the line integral is the difference between the value of $-cu(x,t)$ at the end of the segment and at the beginning of it. In other words:
$$\int_{(x_0,t_0)}^{(0,t_0-x_0/c)} -u_t\,dx - c^2 u_x\,dt = -cc\Big(u\left(0, t_0 - \frac{x_0}{c}\right) - u(x_0, t_0)\Big) = c\,u(x_0, t_0)$$
because of the boundary condition $u(0,t) = 0$.

- Finally there is the segment from $(0, t_0 - x_0/c)$ back to $(ct_0 - x_0, 0)$. Follow the same drill, and parametrize:
$$x = cs \quad \text{and} \quad t = t_0 - \frac{x_0}{c} - s \quad \text{for} \quad 0 \le s \le t_0 - \frac{x_0}{c}$$
and then $dx = c\,ds$ and $dt = -ds$ so the integral becomes
$$\int_{(0,t_0-x_0/c)}^{(ct_0-x_0,0)} -u_t\,dx - c^2 u_x\,dt = \int_0^{t_0-x_0/c} -cu_t\left(cs, t_0 - \frac{x_0}{c} - s\right) + c^2 u_x\left(cs, t_0 - \frac{x_0}{c} - s\right)\,ds$$
And now make the observation that with x and t defined as functions of s as in the parametrization of this segment, we have that
$$\frac{du}{ds} = \frac{\partial u}{\partial x}\frac{dx}{ds} + \frac{\partial u}{\partial t}\frac{dt}{ds} = cu_x\left(cs, t_0 - \frac{x_0}{c} - s\right) - u_t\left(cs, t_0 - \frac{x_0}{c} - s\right).$$
And this is precisely $1/c$ times the integrand, so the value of this part of the line integral is the difference between the value of $cu(x,t)$ at the end of the segment and at the beginning of it. But u is zero at both points, because of the boundary and initial conditions, so that the value of the integral over this segment is zero.

Now we can insert the value of the line integral back into our original computation:
$$\iint_D f(y,s)\,dy\,ds = \iint_D u_{tt} - c^2 u_{xx} = \oint_{\text{bd}(D)} -u_t\,dx - c^2 u_x\,dt = 0 + cu(x_0,t_0) + cu(x_0,t_0) + 0 = 2cu(x_0,t_0).$$
Thus the solution of the initial-boundary value problem is
$$u(x,t) = \frac{1}{2c}\iint_D f(y,s)\,dy\,ds.$$

Page 89, problem 2. The differential equation for the temperature of the rod starting at the moment the ends are immersed is the heat equation $u_t = ku_{xx}$, together with the initial condition $u(x,0) = 1$ for $0 < x < \ell$ and boundary conditions $u(0,t) = u(\ell,t) = 0$ for $t > 0$. This is the initial/boundary-value problem given by equations (13), (14) and (15) on page 87 of the textbook (with $\varphi(x) = 1$ for all x between 0 and ℓ). The solution is given by equation (17):

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-(n\pi/\ell)^2 kt} \sin \frac{n\pi x}{\ell},$$

with the coefficients A_n chosen so that

$$1 = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{\ell}.$$

The problem tells us that

$$A_n = \begin{cases} \dfrac{4}{n\pi} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

So, writing the odd numbers as $2k+1$ for $k = 0, 1, 2, \ldots$ we can express the solution as

$$u(x,t) = \sum_{k=0}^{\infty} \frac{4}{(2k+1)\pi} e^{-[(2k+1)\pi/\ell]^2 kt} \sin \frac{(2k+1)\pi x}{\ell}$$

Page 89, problem 3. The Schrödinger equation looks just like the heat equation with k replaced by $i = \sqrt{-1}$. So the solution of the Dirichlet problem corresponding to it should look just like the solution (equation (17) on page 87) of the heat equation, with k replaced by i:

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-(n\pi/\ell)^2 it} \sin \frac{n\pi x}{\ell}$$

Page 92, problem 1. If we seek separated solutions $u(x,t) = X(x)T(t)$ of the diffusion equation $u_t = ku_{xx}$, we get that $X(x)T'(t) = kX''(x)T(t)$, or in other words

$$\frac{T''(t)}{kT(t)} = \frac{X''(x)}{X(x)} = -\lambda.$$

The boundary conditions $u(0,t) = 0$ and $u_x(\ell,t) = 0$ give us conditions $X(0) = 0$ and $X'(\ell) = 0$ on X, so we solve the equation for X first.

Can we have $\lambda < 0$? In that case, the solution of $X'' + \lambda X = 0$ would be

$$X(x) = c_1 \cosh(\sqrt{-\lambda}\, x) + c_2 \sinh(\sqrt{-\lambda}\, x).$$

The condition $X(0) = 0$ implies that $c_1 = 0$, and then $X'(\ell) = c_2\sqrt{\lambda}\cosh(\sqrt{-\lambda}\,\ell) = 0$ implies that $c_2 = 0$ (since $\cosh x \neq 0$ for all $x \in \mathbb{R}$). So there are no negative eigenvalues.

Can we have $\lambda = 0$? In that case, the solution of $X'' = 0$ is $X(x) = c_1 + c_2 x$, and the boundary conditions again force $c_1 = c_2 = 0$, so zero is not an eigenvalue.

For $\lambda > 0$, the solution of $X'' + \lambda X = 0$ is

$$X(x) = c_1 \cos(\sqrt{\lambda}\, x) + c_2 \sin(\sqrt{\lambda}\, x).$$

The condition $X(0) = 0$ implies that $c_1 = 0$ and then $X'(\ell) = c_2\sqrt{\lambda}\cos(\sqrt{\lambda}\,\ell) = 0$. So we can have $c_2 \neq 0$ if λ is chosen so that

$$\cos(\sqrt{\lambda}\,\ell) = 0\,, \quad \text{in other words} \quad \sqrt{\lambda}\,\ell = \pm\frac{\pi}{2},\ \pm\frac{3\pi}{2},\ \pm\frac{5\pi}{2}, \dots$$

Therefore we have

$$\begin{array}{lll} \text{Eigenvalues:} & \dfrac{(2n+1)^2\pi^2}{4\ell^2} & \text{for } n = 0, 1, 2, 3, \dots \\ \text{Eigenfunctions:} & X_n(x) = \sin\dfrac{(2n+1)\pi x}{2\ell} & \text{for } n = 0, 1, 2, 3, \dots \end{array}$$

Now we deal with the T equation: $T' + \lambda k T = 0$. The solution of this is $T(t) = ce^{-\lambda k t}$. So the solution of the diffusion equation with the given boundary conditions is

$$u(x,t) = \sum_{n=0}^{\infty} A_n e^{-[(2n+1)\pi/2\ell]^2 kt} \sin\frac{(2n+1)\pi x}{2\ell}.$$

Write up solutions of the following to hand in:

- Page 66 (page 64 in the first ed), problems 4, 10
- Page 70 (page 68 in the first ed), problem 2
- Page 79 (page 76 in the first ed), problems 2, 5, 13
- Page 89 (page 87 in the first ed), problems 4, 6
- Page 92 (page 90 in the first ed), problem 2

Page 66, problem 4. Since $u(x,t) = f(x+ct)$ for $x > 0$ and $t < 0$, we have

$$u(x,0) = f(x) \quad \text{and} \quad u_t(x,0) = cf'(x) \quad (\text{for } x < 0)$$

We also have $u_x(0,t) = 0$ because the end is "free". Because of this, we extend f and cf' as *even* functions:

$$\varphi(x) = \begin{cases} f(x) & \text{if } x > 0 \\ f(-x) & \text{if } x < 0 \end{cases} \quad \text{and} \quad \psi(x) = \begin{cases} cf'(x) & \text{if } x > 0 \\ cf'(-x) & \text{if } x < 0 \end{cases}$$

We need to interpret d'Alembert's solution

$$u(x,t) = \frac{1}{2}\Big(\varphi(x+ct) + \varphi(x-ct)\Big) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\, ds$$

in terms of these extended functions. There are two cases: either $x > ct$ or $x < ct$.

In the first case, $x > ct$, we have $x + ct > 0$ and $x - ct > 0$. In this case,

$$\begin{aligned} u(x,t) &= \frac{1}{2}\Big(f(x+ct) + f(x-ct)\Big) + \frac{1}{2c}\int_{x-ct}^{x+ct} cf'(s)\,ds \\ &= \frac{1}{2}\Big(f(x+ct) + f(x-ct)\Big) + \frac{1}{2}\Big(f(x+ct) - f(x-ct)\Big) \\ &= f(x+ct) \end{aligned}$$

In the second case, $x < ct$, we have $x + ct > 0$ and $x - ct < 0$ (or $ct - x > 0$). In this case,

$$\begin{aligned} u(x,t) &= \frac{1}{2}\Big(f(x+ct) + f(ct-x)\Big) + \frac{1}{2c}\left(\int_{x-ct}^{0} cf'(-s)\,ds + \int_{0}^{x+ct} cf'(s)\,ds\right) \\ &= \frac{1}{2}\Big(f(x+ct) + f(ct-x)\Big) + \frac{1}{2}\left(\int_{ct-x}^{0} -f'(q)\,dq + f(x+ct) - f(0)\right) \\ &= \frac{1}{2}\Big(f(x+ct) + f(ct-x)\Big) + \frac{1}{2}\Big(f(ct-x) - f(0) + f(x+ct) - f(0)\Big) \\ &= f(x+ct) + f(ct-x) - f(0) \end{aligned}$$

where we made the substitution $q = -s$ in one of the integrals. We conclude that

$$u(x,t) = \begin{cases} f(x+ct) & \text{if } x \geq ct \\ f(x+ct) + f(ct-x) - f(0) & \text{if } x \leq ct \end{cases}$$

Page 66, problem 10. We need to solve $u_{tt} = 9u_{xx}$ for $0 < x < \pi/2$ (so $c = 3$) with initial conditions $u(x,0) = \cos x$ and $u_t(x,0) = 0$ on that interval, and $u_x(0,t) = 0$ (so the left end is "free") and $u(\pi/2\,,\,t) = 0$ (so the right end is "fixed"). Since $\cos(-x) = \cos x$ and $\cos(\pi/2 + x) = -\cos(\pi/2 - x)$, the appropriate periodic extension of $u(x,0)$ — which should be even at multiples of π and odd at odd multiples of $\pi/2$ — is just $\cos x$. So we can use d'Alembert's solution with $\varphi(x) = \cos x$ for all x and $\psi(x) = 0$ and the solution is

$$u(x,t) = \frac{1}{2}\Big(\cos(x+3t) + \cos(x-3t)\Big) = \cos x \cos 3t$$

after using the trig identities $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$.

Page 70, problem 2. We need to solve $v_t - kv_{xx} = f(x,t)$ for $x > 0$ and $t > 0$ together with the initial condition $v(x,0) = \varphi(x)$ and the boundary condition $v(0,t) = h(t)$. So to start, we suppress the boundary condition by letting

$$w(x,t) = v(x,t) - h(t).$$

Then w satisfies $w_t - kw_{xx} = f(x,t) - h'(t)$ for $x > 0$ and $t > 0$ together with the initial condition $w(x,0) = \varphi(x) - h(0)$ and the boundary condition $w(0,t) = 0$. We are going to need the odd extension of $w(x,0)$, which we're going to call $Q(x)$:

$$Q(x) = \begin{cases} \varphi(x) - h(0) & \text{if } x > 0 \\ -\varphi(-x) + h(0) & \text{if } x < 0 \end{cases}$$

Also, let $F(x,t)$ be the odd extension of $f(x,t) - h'(t)$, so

$$F(x,t) = \begin{cases} f(x,t) - h'(t) & \text{if } x > 0 \\ -f(-x,t) + h'(t) & \text{if } x < 0 \end{cases}$$

Then

$$\begin{aligned}
w(x,t) &= \int_{-\infty}^{\infty} S(x-y,t)Q(y)\,dy + \int_0^t \int_{-\infty}^{\infty} S(x-y,t-s)F(y,s)\,dy\,ds \\
&= \int_0^{\infty} S(x-y,t)\Big(\varphi(y) - h(0)\Big)\,dy + \int_{-\infty}^{0} S(x-y,t)\Big(-\varphi(-y) + h(0)\Big)\,dy \\
&\quad + \int_0^t \int_0^{\infty} S(x-y,t-s)\Big(f(y,s) - h'(s)\Big)\,dy\,ds + \int_0^t \int_{-\infty}^{0} S(x-y,t-s)\Big(-f(-y,s) + h'(s)\Big)\,dy\,ds \\
&= \int_0^{\infty} S(x-y,t)\Big(\varphi(y) - h(0)\Big)\,dy + \int_0^{\infty} S(x+y,t)\Big(-\varphi(y) + h(0)\Big)\,dy \\
&\quad + \int_0^t \int_0^{\infty} S(x-y,t-s)\Big(f(y,s) - h'(s)\Big)\,dy\,ds + \int_0^t \int_0^{\infty} S(x+y,t-s)\Big(-f(y,s) + h'(s)\Big)\,dy\,ds \\
&= \int_0^{\infty} \Big(S(x-y,t) - S(x+y,t)\Big)\Big(\varphi(y) - h(0)\Big)\,dy \\
&\qquad + \int_0^t \int_0^{\infty} \Big(S(x-y,t-s) - S(x+y,t-s)\Big)\Big(f(y,s) - h'(s)\Big)\,dy\,ds
\end{aligned}$$

There's only a bit of simplification (or $\mathcal{E}$rf-ificaion) we can do here. The terms with $h(0)$ and $h'(s)$ can be brought out of at least one integral and then we need to compute an integral of the form:

$$\int_0^{\infty} S(x+y,t) - S(x-y,t)\,dy = \int_0^{\infty} \frac{1}{\sqrt{4\pi kt}} \left(e^{-\frac{(x+y)^2}{4kt}} - e^{-\frac{(x-y)^2}{4kt}} \right) dy.$$

Make the change of variables $p = (x+y)/\sqrt{4kt}$ (so $dp = dy/\sqrt{4kt}$) in the first integral and $q = (x-y)/\sqrt{4kt}$ (so $dq = -dy/\sqrt{4kt}$) in the second to get

$$\int_{x/\sqrt{4kt}}^{\infty} \frac{1}{\sqrt{\pi}} e^{-p^2}\,dp - \int_{-\infty}^{x/\sqrt{4kt}} \frac{1}{\sqrt{\pi}} e^{-q^2}\,dq = \left[\frac{1}{2} - \frac{1}{2}\,\mathcal{E}\mathrm{rf}\left(\frac{x}{\sqrt{4kt}}\right)\right] - \left[\frac{1}{2} + \frac{1}{2}\,\mathcal{E}\mathrm{rf}\left(\frac{x}{\sqrt{4kt}}\right)\right] = -\,\mathcal{E}\mathrm{rf}\left(\frac{x}{\sqrt{4kt}}\right).$$

Using this, we can rewrite our expression for $w(x,t)$ above as

$$\begin{aligned}
w(x,t) &= \frac{1}{\sqrt{4\pi kt}} \int_0^{\infty} \left(e^{-\frac{(x-y)^2}{4kt}} - e^{-\frac{(x+y)^2}{4kt}} \right) \varphi(y)\,dy - h(0)\,\mathcal{E}\mathrm{rf}\left(\frac{x}{\sqrt{4kt}}\right) \\
&\quad + \int_0^t \int_0^{\infty} \frac{1}{\sqrt{4\pi k(t-s)}} \left(e^{-\frac{(x-y)^2}{4k(t-s)}} - e^{-\frac{(x+y)^2}{4k(t-s)}} \right) f(y,s)\,dy\,ds - \int_0^t h'(s)\,\mathcal{E}\mathrm{rf}\left(\frac{x}{\sqrt{4k(t-s)}}\right) ds
\end{aligned}$$

And finally (remember, we were solving for $v(x,t)$!)

$$\begin{aligned}
v(x,t) &= h(t) + \frac{1}{\sqrt{4\pi kt}} \int_0^{\infty} \left(e^{-\frac{(x-y)^2}{4kt}} - e^{-\frac{(x+y)^2}{4kt}} \right) \varphi(y)\,dy - h(0)\,\mathcal{E}\mathrm{rf}\left(\frac{x}{\sqrt{4kt}}\right) \\
&\quad + \int_0^t \int_0^{\infty} \frac{1}{\sqrt{4\pi k(t-s)}} \left(e^{-\frac{(x-y)^2}{4k(t-s)}} - e^{-\frac{(x+y)^2}{4k(t-s)}} \right) f(y,s)\,dy\,ds - \int_0^t h'(s)\,\mathcal{E}\mathrm{rf}\left(\frac{x}{\sqrt{4k(t-s)}}\right) ds
\end{aligned}$$

Page 79, problem 2. We need to solve $u_{tt} = c^2 u_{xx} + e^{ax}$ on the whole line with zero initial position and velocity. By the Duhamel formula (only the term for the inhomogeneity of the differential equation will be present):

$$\begin{aligned} u(x,t) &= \frac{1}{2c}\int_0^t \int_{x-ct+cs}^{x+ct-cs} e^{ay}\, dy\, ds = \frac{1}{2ac} e_0^t e^{a(x+ct-cs)} - e^{a(x-ct+cs)}\, ds \\ &= -\frac{1}{2a^2c^2}\left[e^{a(x+ct-cs)} - e^{a(x-ct+cs)}\right]_0^t = -\frac{1}{2a^2c^2}\left[2e^{ax} - a^{a(x+ct)} - e^{a(x-ct)}\right] \\ &= \frac{1}{2a^2c^2}\left[a^{a(x+ct)} + e^{a(x-ct)} - 2e^{ax}\right] \end{aligned}$$

Page 79, problem 5. We need to verify that the solution of

$$u_{tt} = c^2 u_{xx} + f\,, \qquad u(x,0) = 0\,, \qquad u_t(x,0) = 0$$

on the whole line is given by

$$u(x,t) = \frac{1}{2c}\int_0^t \int_{x-ct+cs}^{x+ct-cs} f(y,s)\, dy\, ds.$$

It's easy to see that

$$u(x,0) = \frac{1}{2c}\int_0^0 \cdots = 0$$

so that's one initial condition out of the way. Next, calculate

$$\frac{\partial u}{\partial t} = \frac{1}{2c}\int_x^x f(y,t)\, dy + \frac{1}{2c}\int_0^t \frac{\partial}{\partial t}\left(\int_{x-ct+cs}^{x+ct-cs} f(y,s)\, dy\right) ds$$

using Leibniz's rule (or the fundamental theorem of calculus in its fanciest form). The first term is clearly zero. Use Leibniz again on the second term (the only $t's$ are in the limits of integration) to conclude that:

$$\frac{\partial u}{\partial t} = \frac{1}{2}\int_0^t f(x+ct-cs,s) + f(x-ct+cs,s)\, ds.$$

Observe that this shows that $u_t(x,0) = 0$ since we'll have another integral from 0 to 0.

Next, use Leibniz on the expression for u_t above to get

$$\frac{\partial^2 u}{\partial t^2} = f(x,t) + \frac{c}{2}\int_0^t \frac{\partial f}{\partial x}(x+ct-cs,s) - \frac{\partial f}{\partial x}(x-ct+cs,s)\, ds$$

because the only t's in the integrand occur in the x slot of f.

Now we move on to the x-derivatives of u:

$$\frac{\partial u}{\partial x} = \frac{1}{2c}\int_0^t f(x+ct-cs,s) - f(x-ct+cs,s)\, ds$$

since the only place om the formula for u that x appears is in the limits of integration of the inner integral. And then

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{2c}\int_0^t \frac{\partial f}{\partial x}(x+ct-cs,s) - \frac{\partial f}{\partial x}(x-ct+cs,s)\,ds$$

And comparing the expressions for u_{tt} ad u_{xx} above, you can see that $u_{tt} - c^2 u_{xx} = f(x,t)$.

Page 79, problem 13. We need to solve the wave equation $u_{tt} = c^2 u_{xx}$ for $x > 0$ with initial conditions $u(x,0) = x$ and $u_t(x,0) = 0$ and boundary condition $u(0,t) = t^2$.

We will trade the inhomogeneous boundary condition for an inhomogeneous equation by letting $v(x,t) = u(x,t) - t^2$. Then v will satisfy

$$v_{tt} = c^2 v_{xx} - 2 \quad \text{for } x > 0$$

together with initial conditions

$$v(x,0) = x \quad \text{and} \quad v_t(x,0) = 0$$

and boundary condition

$$v(0,t) = 0.$$

Because the boundary condition is a Dirichlet condition, we need to extend the initial data and the inhomogeneity in the equation to be odd functions in x. Fortunately, the initial data are already odd functions. To work on the whole line we need to rewrite the differential equation is

$$v_{tt} = c^2 v_{xx} + f(x) \qquad \text{where} \qquad f(x) = \begin{cases} -2 & \text{for } x > 0 \\ 2 & \text{for } x < 0 \end{cases}$$

Then

$$v(x,t) = \frac{1}{2c}\int_0^t \int_{x-ct}^{x+ct} f(y)\,dy\,ds + \frac{1}{2}\Big[(x+ct) + (x-ct)\Big]$$

If $x > ct$ then $f(y)$ is identically -2 over the triangular domain D of integration, which has area $\frac{1}{2}(2ct)t = ct^2$, so for $x > ct$ we have $v(x,t) = -t^2 + x$.

For $x < ct$, the function $f(x)$ is equal to $+2$ over the small triangular portion of D with vertices $(0,0)$, $\left(0, t - \dfrac{x}{c}\right)$, $(x - ct, 0)$, which has area

$$\frac{1}{2}(ct - x)\left(t - \frac{x}{c}\right) = \frac{1}{2c}(c^2t^2 - 2ctx + x^2)$$

So the integral

$$\int_0^t \int_{x-ct}^{x+ct} f(y)\,dy\,ds = -2ct^2 + 4\left(\frac{1}{2c}(c^2t^2 - 2ctx + x^2)\right) = \frac{2x^2}{c} - 4tx.$$

Therefore

$$v(x,t) = \begin{cases} x - t^2 & \text{for } x > ct \\ x + \dfrac{x^2}{c^2} - \dfrac{2tx}{c} & \text{for } x < ct \end{cases}$$

Since $u(x,t) = v(x,t) + t^2$, we have

$$u(x,t) = \begin{cases} x & \text{for } x > ct \\ x + \dfrac{x^2}{c^2} - \dfrac{2tx}{c} + t^2 & \text{for } x < ct \end{cases}$$

Page 89, problem 4. We want to find "separated solutions" for the "damped wave equation"

$$u_{tt} = c^2 u_{xx} - ru_t \qquad \text{for } 0 < x < \ell \text{ and } t > 0$$

(where $0 < r < \dfrac{2\pi c}{\ell}$) that satisfy the initial conditions

$$u(x,0) = \varphi(x) \quad \text{and} \quad u_t(x,0) = \psi(x)$$

and the boundary conditions

$$u(0,t) = 0 \quad \text{and} \quad u(\ell,t) = 0$$

Substitute $u(x,t) = X(x)T(t)$ into the equation to obtain

$$XT'' + rXT' = c^2X''T$$

or

$$\frac{T'' + rT'}{c^2T} = \frac{X''}{X} = -\lambda$$

So the problem for $X(x)$ is

$$X'' + \lambda X = 0 \quad \text{and} \quad X(0) = X(\ell) = 0.$$

This tells us that

$$\lambda = \frac{n^2\pi^2}{\ell^2} \qquad \text{and} \qquad X_n(x) = \sin\frac{n\pi x}{\ell}\,,\ n = 1, 2, \ldots$$

Now $T(t)$ must be a solution of

$$T'' + rT' + \lambda c^2 T = 0 \qquad \text{or} \qquad T'' + rT + \frac{n^2c^2\pi^2}{\ell^2}T = 0.$$

The roots of the characteristic equation are

$$-\frac{r}{2} \pm \frac{1}{2}\sqrt{r^2 - \frac{4n^2c^2\pi^2}{\ell^2}}$$

But the quantity under the radical is negative for $n = 1, 2, 3, \ldots$, since $0 < r < \dfrac{2\pi c}{\ell}$, so we rewrite it as

$$-\frac{r}{2} \pm i\frac{1}{2}\sqrt{\frac{4n^2c^2\pi^2}{\ell^2} - r^2}$$

and so

$$T_n = A_n e^{-rt/2}\cos\left(\sqrt{\frac{4n^2\pi^2c^2}{\ell^2} - r^2}\,\frac{t}{2}\right) + B_n e^{-rt/2}\sin\left(\sqrt{\frac{4n^2\pi^2c^2}{\ell^2} - r^2}\,\frac{t}{2}\right)$$

and so the solution of the initial/boundary-value problem is

$$u(x,t) = \sum_{n=1}^{\infty} X_n(x)T_n(t) = \sum_{n=1}^{\infty}\left[A_n e^{-rt/2}\cos\left(\sqrt{\frac{4n^2\pi^2c^2}{\ell^2} - r^2}\,\frac{t}{2}\right) + B_n e^{-rt/2}\sin\left(\sqrt{\frac{4n^2\pi^2c^2}{\ell^2} - r^2}\,\frac{t}{2}\right)\right]\sin\frac{n\pi x}{\ell}$$

where

$$\sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{\ell} = \varphi(x)$$

and

$$\sum_{n=1}^{\infty} \left(\frac{1}{2} \sqrt{\frac{4n^2\pi^2c^2}{\ell^2} - r^2}\, B_n - \frac{r}{2} A_n \right) \sin \frac{n\pi x}{\ell} = \psi(x).$$

Page 89, problem 6. We want to find separated solutions to the equation

$$tu_t = u_{xx} + 2u \qquad \text{for } 0 < x < \pi \text{ and } t > 0$$

that satisfy the initial condition $u(x, 0) = 0$ and the boundary conditions $u(0, t) = u(\pi, t) = 0$. Usually, we'd expect there not to be any non-zero solution to this problem, but the t multiplying the u_t makes something odd happen for $t = 0$, so specifying the initial condition there is likely to cause something unexpected. Substitute $u(x, t) = X(x)T(t)$ into the equation to obtain

$$tXT' = X''T + 2XT$$

or

$$\frac{tT'}{T} - 2 = \frac{X''}{X} = -\lambda$$

So the problem for $X(x)$ is

$$X'' + \lambda X = 0 \quad \text{and} \quad X(0) = X(\pi) = 0.$$

This tells us that

$$\lambda = n^2 \qquad \text{and} \qquad X_n(x) = \sin nx\,,\ n = 1, 2, \ldots$$

With $\lambda = n^2$, the equation for T becomes

$$\frac{tT'}{T} - 2 = -n^2 \qquad \text{or} \qquad tT' = (2 - n^2)T$$

which is a separable (in the ODE sense) first-order equation so

$$\frac{dT}{(2 - n^2)T} = \frac{dt}{t} \qquad \text{so} \frac{1}{2 - n^2} \ln T = \ln t + \ln c$$

which give us that

$$T_n(t) = c_n t^{2-n^2}.$$

For $n = 1$, $T_1(t) = c_1 t$ and for *any* value of c_1, the product

$$X_1(x)T_1(t) = c_1 t \sin x$$

satisfies all the conditions of the problem. Therefore, there are infinitely many solutions.

Page 92, problem 2. We want to solve the wave equation $u_{tt} = c^2 u_{xx}$ for $t > 0$ and $0 < x < \ell$ with boundary conditions $u_x(0, t) = 0$ and $u(\ell, t) = 0$, so the string is fixed at its right end and free at its left. So we do the usual separation of variables and obtain:

$$\frac{T''}{c^2T} = \frac{X''}{X} = \lambda \qquad \text{and} \qquad X'(0) = X(\ell) = 0$$

(we'll be "naïve" here and pretend we don't know that λ is not going to be negative, just to practice proving that it can't be).

We need to find the eigenvalues of $X'' - \lambda X = 0$ with $X'(0) = X(\ell) = 0$. Since the solution is different in this case, we check the possibility of $\lambda = 0$ first: If $\lambda = 0$ the $X'' = 0$, so $X = ax + b$. Then $X'(0) = 0$ tells us that $a = 0$ and $X(\ell) = 0$ tells us that $b = 0$, so zero is *not* an eigenvalue for this problem.

For $\lambda \neq 0$ we have that $X = c_1 e^{\sqrt{\lambda}\,x} + c_2 e^{-\sqrt{\lambda}\,x}$. Take the derivative of this and set $x = 0$ to get $X'(0) = \sqrt{\lambda}\,c_1 - \sqrt{\lambda}\,c_2 = 0$. Since we're assuming $\lambda \neq 0$, this implies that $c_1 = c_2$, so let's call their common value c so that at this point, $X = c(e^{\sqrt{\lambda}\,x} + e^{-\sqrt{\lambda}\,x})$. Next, set $x = \ell$ and we get that $c(e^{\sqrt{\lambda}\,\ell} + c_2 e^{-\sqrt{\lambda}\,\ell}) = 0$, in other words (assuming $c \neq 0$ so that we really have an eigenfunction)

$$e^{\sqrt{\lambda}\,\ell} = -\frac{1}{e^{\sqrt{\lambda}\,\ell}} \quad \text{which implies} \quad e^{2\sqrt{\lambda}\,\ell} = -1$$

which means that $2\sqrt{\lambda}\,\ell$ must be an *odd* multiple of πi. Therefore

$$\sqrt{\lambda} = \frac{(2n+1)\pi i}{2\ell} \quad \text{which implies} \quad X_n(x) = c_n \cos\left(\frac{2n+1}{2\ell}\pi x\right) \quad n = 0, 1, \ldots$$

And with $\lambda = -(2n+1)^2\pi^2/(2\ell)^2$, we can solve the T equation and get

$$T_n(t) = a_n \cos\left(\frac{2n+1}{2\ell}\pi ct\right) + b_n \left(\frac{2n+1}{2\ell}\pi ct\right).$$

Now let $A_n = c_n a_n$ and $B_n = c_n b_n$ and we can write the solution as

$$u(x,t) = \sum_{n=0}^{\infty} \left[A_n \cos\left(\frac{2n+1}{2\ell}\pi ct\right) + B_n \sin\left(\frac{2n+1}{2\ell}\pi ct\right)\right] \cos\left(\frac{2n+1}{2\ell}\pi x\right)$$

which awaits determination of the A_n and B_n using initial conditions.