Math 425 / AMCS 525 **Dr. DeTurck**
Assignment 6 **Due Thursday, March 17, 2016**

Topics for this week — Fourier series solutions – how to find them for various kinds of boundary conditions, perhaps a little about convergence

Sixth Homework Assignment - due Thursday, March 17

Reading: Read sections 4.3, 5.1, 5.2 and 5.3 of the text

Be prepared to discuss the following problems in class:

- Page 100 (page 97 in the first ed), problems 2, 6, 16
- Page 111 (page 107 in the first ed), problems 3, 4
- Page 117 (page 113 in the first ed), problems 2, 8
- Page 122 (page 118 in the first ed), problems 3, 5, 12

Page 100, problem 2. (a) We have $X'' + \lambda X = 0$, so if $\lambda = 0$ then $X'' = 0$ in which case $X = c_1 + c_2 x$. Then the boundary condition $X'(0) - a_0 X(0) = 0$ implies that $c_2 - a_0 c_1 = 0$, i.e., $c_2 = a_0 c_1$ so now we have $X = c_1(1 + a_0 x)$. And so

$$X'(\ell) + a_\ell X(\ell) = c_1 a_0 + a_\ell c_1 (1 + a_0 \ell) = c_1(a_0 + a_\ell + a_0 a_\ell).$$

In order that $c_1 \neq 0$, we must have $a_0 + a_\ell + a_0 a_\ell = 0$, which gives the condition in the problem.

(b) As noted in part (a), if the condition is satisfied, then $X = c(1 + a_0 x)$.

Page 100, problem 6. (a) The problem is $X'' + \lambda X = 0$. If we're looking for negative eigenvalues, then let $\lambda = -\beta^2$ so the problem becomes $X'' - \beta^2 X = 0$, the general solution of which is

$$X = c_1 e^{\beta x} + c_2 e^{-\beta x}.$$

The boundary conditions are

$$X'(0) - aX(0) = 0 \qquad \text{and} \qquad X'(\ell) + aX(\ell) = 0$$

which become (on substitution of the solution above and noting that $X' = \beta c_1 e^{\beta x} - \beta c_2 e^{-\beta x}$)

$$\beta(c_1 - c_2) - a(c_1 + c_2) = 0 \qquad \text{and} \qquad \beta(c_1 e^{\beta\ell} - c_2 e^{-\beta\ell}) + a(c_1 e^{\beta\ell} + c_2 e^{-\beta\ell})$$

in other words

$$(\beta - a)c_1 - (\beta + a)c_2 = 0 \qquad \text{and} \qquad e^{\beta\ell}(\beta + a)c_1 + e^{-\beta\ell}(a - \beta)c_2$$

If this system of two equations has a non-trivial solution for c_1 and c_2 then we must have

$$0 = \det \begin{bmatrix} \beta - a & -\beta - a \\ e^{\beta\ell}(\beta + a) & e^{-\beta\ell}(-\beta + a) \end{bmatrix} = e^{\beta\ell}(\beta + a)^2 - e^{-\beta\ell}(\beta - a)^2$$

Put one of the terms on the other side of the equation and cross-multiply and obtain:

$$e^{2\beta\ell} = \frac{(\beta - a)^2}{(\beta + a)^2}$$

and we need to solve this for β. First, notice that $\beta = 0$ is always a solution of this equation, since both sides equal 1 there, but we're interested only in non-zero solutions. Next, the fraction $F(\beta)$ on the right side is always positive, has a vertical asymptote at $\beta = -a$, is equal to zero at $\beta = a$, and approaches 1 as $\beta \to \pm\infty$. These might lead us to the observation that

$$F(-\beta) = \frac{1}{F(\beta)} \qquad \text{just as} \qquad e^{-2\beta\ell} = \frac{1}{e^{2\beta\ell}}$$

So we need only consider either $\beta > 0$ or $\beta < 0$, since either $\pm\beta$ will give us the same eigenvalue $\lambda = -\beta^2$.

If $a > 0$, then $e^{2\beta\ell} > 1$ for all $\beta > 0$ but $-1 < F(\beta) < 1$ for $\beta > 0$, so there are no positive solutions, and therefore no solutions (other than $\beta = 0$). So there are no negative eigenvalues if $a > 0$.

If $a < 0$ then the vertical asymptote is to the right of the vertical axis, so the part of graph of $F(\beta)$ to the right of the asymptote, where $F(\beta)$ decreases from $+\infty$ to its limiting value of 1, will intersect the graph of $e^{2\beta\ell}$ once, and give us one negative eigenvalue. The question is whether the part of the graph of $F(x)$ to the left of the asymptote but to the right of the vertical axis will intersect the exponential graph on its way up from 1 to $+\infty$. An intersection will occur if $F'(0) < 2\ell$, since if this is the case then the graph of $F(x)$ will start out below the exponential curve for small positive values of β, and so the graphs must cross as value of $F(x)$ heads toward $+\infty$. So we calculate

$$F'(\beta) = \frac{(\beta + a)^2[2(\beta - a)] - (\beta - a)^2[2(\beta + a)]}{(\beta + a)^4} = \frac{2(\beta - a)[(\beta + a) - (\beta - a)]}{(\beta + a)^3} = \frac{4a(\beta - a)}{(\beta + a)^3}.$$

Therefore $F'(0) = -4/a$. The graph of $F(\beta)$ and the exponential graph will be just tangent at $\beta = 0$ if $-4/a = 2\ell$, in other words if $a = -2/\ell$. And for $0 > a \geq -2/\ell$ the two graphs will not cross for $0 < \beta < -a$, whereas if $a < -2/\ell$ there will be a solution between 0 and $-a$. Therefore we can conclude that there is only one negative eigenvalue if $0 > a > -2/\ell$ and two negative eigenvalues if $a < -2/\ell$.

(b) According to problem 2(a), zero will be an eigenvalue if and only if $a + a = -a^2\ell$, in other words $\ell a^2 + 2a = a(\ell a + 2) = 0$, the solutions of which are $a = 0$ and $a = -2/\ell$.

Page 100, problem 16. Even though the problem asks only about the positive eigenvalues, we'll show that there are no other eigenvalues as well. The eigenvalue problem is $X'''' + \lambda X = 0$, so the characteristic polynomial is $t^4 + \lambda = 0$. If $\lambda = re^{i\theta}$ then the roots of the characteristic polynomial are $r^{1/4}e^{i\theta/4}$, $ir^{1/4}e^{i\theta/4}$, $-r^{1/4}e^{i\theta/4}$ and $-ir^{1/4}e^{i\theta/4}$. Setting $\beta = r^{1/4}e^{i\theta/4}$, we'll write the roots as β, $i\beta$, $-\beta$ and $-i\beta$. So we have

$$X = c_1e^{\beta x} + c_2e^{i\beta x} + c_3e^{-\beta x} + c_4e^{-i\beta x}$$

and then

$$X'' = \beta^2c_1e^{\beta x} - \beta^2c_2e^{i\beta x} + \beta^2c_3e^{-\beta x} - \beta^2c_4e^{-i\beta x}.$$

The conditions $X(0) = 0$ and $X''(0) = 0$ translate into

$$c_1 + c_2 + c_3 + c_4 = 0 \quad \text{and} \quad \beta^2(c_1 - c_2 + c_3 - c_4) = 0$$

which tell us that $c_1 + c_3 = 0$ and $c_2 + c_4 = 0$ (because the sum and difference of these two quantities are both zero). So let $c_3 = -c_1$ and $c_4 = -c_2$. Using that $\sinh x = \frac{1}{2}(e^x - e^{-x})$ and $\sin x = -i\frac{1}{2}(e^{ix} - e^{-ix})$, we can rewrite X as:

$$X(x) = c_1(e^{\beta x} - e^{-\beta x}) + c_2(e^{i\beta x} - e^{-i\beta x}) = a \sinh \beta x + b \sin \beta x$$

where $a = 2c_1$ and $b = 2ic_2$. Then the conditions $X(\ell) = 0$ and $X''(\ell) = 0$ translate into

$$a \sinh \beta\ell + b \sin \beta\ell = 0 \quad \text{and} \quad \beta^2(a \sinh \beta\ell - b \sin \beta\ell) = 0$$

So we have the system of equations for a and b:

$$\begin{bmatrix} \sinh \beta\ell & \sin \beta\ell \\ \sinh \beta\ell & \sin \beta\ell \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The determinant of the matrix is $-2 \sinh \beta\ell \sin \beta\ell$, so we have to consider values of β for which $\sin \beta\ell = 0$ and for which $\sinh \beta\ell = 0$.

So let's take a moment to analyze each of the two possibilities. First, consider the equation $\sinh z = 0$ for $z \in \mathbb{C}$. Writing $z = u + iv$, with $u, v \in \mathbb{R}$, we can express:

$$\begin{aligned}
\sinh z = \sinh(u + iv) &= \frac{1}{2}(e^{u+iv} - e^{-(u+iv)}) = \frac{1}{2}(e^u e^{iv} - e^{-u} e^{-iv}) \\
&= \frac{1}{2}[e^u \cos v + ie^u \sin v - e^{-u} \cos v + ie^{-u} \sin v] \\
&= \frac{1}{2}\left[\cos v(e^u - e^{-u}) + i \sin v(e^u + e^{-u})\right] \\
&= \sinh u \cos v + i \cosh u \sin v
\end{aligned}$$

So $\sinh z = 0$ implies that both $\sinh u \cos v = 0$ and $\cosh u \sin v = 0$. Since $\cosh u$ is never zero for $u \in \mathbb{R}$, the latter equation implies that $v = n\pi$ for $n \in \mathbb{Z}$. Putting this into the first equation makes $\cos v = \pm 1$ so we also need $\sinh u = 0$, in other words $u = 0$. So the only complex roots of $\sinh z = 0$ are $z = n\pi i$ for $n \in \mathbb{Z}$.

Likewise,

$$\begin{aligned}
\sin z = \sin(u + iv) &= \frac{1}{2i}(e^{i(u+iv)} - e^{-i(u+iv)}) = \frac{1}{2i}(e^{-v+iu} - e^{v-iu}) \\
&= \frac{1}{2i}[e^{-v} \cos u + ie^{-v} \sin u - e^v \cos u + ie^v \sin u] \\
&= \frac{1}{2i}\left[\cos u(e^{-v} - e^v) + i \sin u(e^v + e^{-v})\right] \\
&= \sin u \left(\frac{e^v + e^{-v}}{2}\right) + i \cos u \left(\frac{e^v - e^{-v}}{2}\right) \\
&= \sin u \cosh v + i \cos u \sinh v
\end{aligned}$$

So $\sin z = 0$ implies that both $\sin u \cosh v = 0$ and $\cos u \sinh v = 0$. Since $\cosh v$ is never zero for $v \in \mathbb{R}$, the former equation implies that $u = n\pi$ for $n \in \mathbb{Z}$. Putting this into the second equation makes $\cos u = \pm 1$ so we also need $\sinh v = 0$, in other words $v = 0$. So the only complex roots of $\sin z = 0$ are $z = n\pi$ for $n \in \mathbb{Z}$, in other words, the ones we already knew about.

The conclusion of all this is that the only possible values of β are those for which $\beta\ell = n\pi$ or $\beta\ell = n\pi i$. In either case we get that

$$\lambda = \frac{n^4\pi^4}{\ell^4} \quad \text{with eigenfunction} \quad X = sin\frac{n\pi x}{\ell} \quad \text{for} \quad n = 1, 2, 3, \dots$$

Using negative or imaginary values of n would be redundant, and you can check that $\lambda = 0$ is *not* an eigenvalue.

Page 111, problem 3. If $\varphi(x) = x$ on $0 < x < \ell$ then

(a) For the Fourier sine series we have

$$x \sim \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{\ell}\right)$$

where

$$B_n = \frac{2}{\ell}\int_0^\ell x \sin\left(\frac{n\pi x}{\ell}\right)\, dx$$

Integrate by parts with $u = x$ and $dv = \sin\left(\frac{n\pi x}{\ell}\right)$, so $du = dx$ and $v = -\frac{\ell}{n\pi}\cos\left(\frac{n\pi x}{\ell}\right)$ to get

$$B_n = \frac{2}{\ell}\left[-\frac{\ell x}{n\pi}\cos\left(\frac{n\pi x}{\ell}\right) + \frac{\ell^2}{n^2\pi^2}\sin\left(\frac{n\pi x}{\ell}\right)\right]_0^\ell = \frac{(-1)^{n+1}2\ell}{n\pi}$$

So

$$x \sim \sum_{n=0}^{\infty} \frac{(-1)^{n+1}2\ell}{n\pi}\sin\left(\frac{n\pi x}{\ell}\right).$$

(b) For the Fourier cosine series we have

$$x \sim \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{\ell}\right)$$

where

$$A_n = \frac{2}{\ell}\int_0^\ell x \sin\left(\frac{n\pi x}{\ell}\right)\, dx.$$

So

$$A_0 = \frac{2}{\ell}\int_0^\ell x\, dx = \ell$$

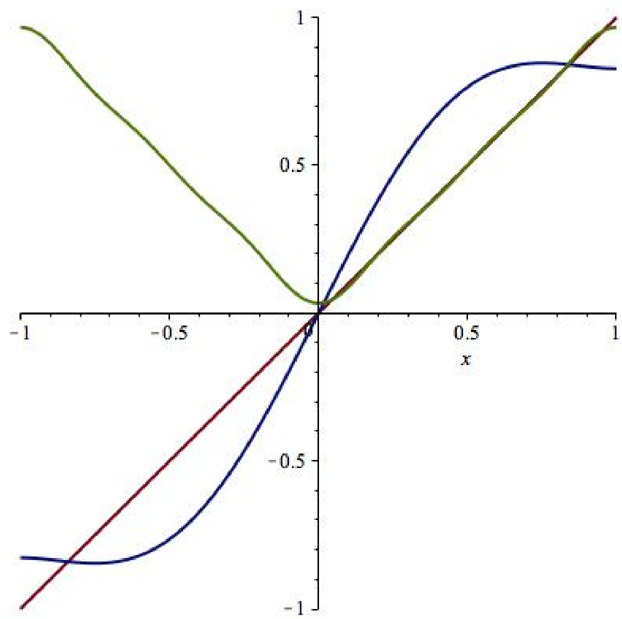
and to find the other A_n, integrate by parts with $u = x$ and $dv = \cos\left(\frac{n\pi x}{\ell}\right)$, so $du = dx$ and $v = \frac{\ell}{n\pi}\sin\left(\frac{n\pi x}{\ell}\right)$ to get

$$A_n = \frac{2}{\ell}\left[\frac{\ell x}{n\pi}\sin\left(\frac{n\pi x}{\ell}\right) + \frac{\ell^2}{n^2\pi^2}\cos\left(\frac{n\pi x}{\ell}\right)\right]_0^\ell = \frac{[(-1)^n - 1]2\ell}{n^2\pi^2}$$

(after making the observation that $\cos n\pi = (-1)^n$). So the even-numbered terms are all zero, and the odd-numbered terms have an additional factor of -2 in the numerator. Writing $n = 2k + 1$, we get that

$$x \sim \frac{\ell}{2} - \sum_{k=0}^{\infty} \frac{4\ell}{(2k+1)^2\pi^2}\cos\left(\frac{(2k+1)\pi x}{\ell}\right).$$

Here's a graph (with $\ell = 1$) of the function x on the interval $[-\ell, \ell]$ (in red), the sum of the first three terms of the sine series (in blue) and the sum of the first three non-zero terms of the cosine series (in green).



The reason the cosine series does a better job of tracking $y = x$ on the interval from 0 to ℓ is that the even periodic extension of $y = x$ is a continuous function whereas the odd periodic extension is not, and the Fourier series of a continuous function converges faster than that of a discontinuous one.

Page 111, problem 4. Since $|\sin x|$ is already an even periodic function, we can find its Fourier cosine series on the interval $[0, \pi]$, where it is equal to $\sin x$. So we'll have

$$|\sin x| \sim \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos nx$$

where

$$\begin{aligned} A_n &= \frac{2}{\pi} \int_0^\pi \sin x \cos nx \, dx = \frac{1}{\pi} \int_0^\pi \sin(n+1)x - \sin(n-1)x \, dx \\ &= \frac{1}{\pi} \left[\frac{1}{n-1} \cos(n-1)x - \frac{1}{n+1} \cos(n+1)x \right]_0^\pi \\ &= \frac{1}{\pi} [(-1)^{n-1} - 1] \left[\frac{1}{n-1} - \frac{1}{n+1} \right] \\ &= \begin{cases} 0 & \text{if } n \text{ is odd} \\ -\dfrac{4}{\pi(n^2-1)} & \text{if } n \text{ is even} \end{cases} \end{aligned}$$

(the formula even holds for $n=0$). Therefore (putting $n=2k$)

$$|\sin x| \sim \frac{2}{\pi} - \sum_{k=1}^{\infty} \frac{4}{\pi(4k^2-1)} \cos 2kx.$$

If we put $x=0$ then

$$0 = \frac{2}{\pi} - \sum_{k=1}^{\infty} \frac{4}{\pi(4k^2-1)}$$

since all the cosines are equal to 1 for $x=0$. We can rearrange this to get

$$\sum_{k=1}^{\infty} \frac{1}{4k^2-1} = \frac{1}{2}.$$

Likewise, put $x=\pi/2$ to get

$$1 = \frac{2}{\pi} - \sum_{k=1}^{\infty} \frac{(-1)^n 4}{\pi(4k^2-1)}$$

which can be rearranged to get

$$\sum_{k=1}^{\infty} \frac{(-1)^n}{4k^2-1} = \frac{1}{2} - \frac{\pi}{4}$$

Page 117, problem 2. Suppose $\alpha = \frac{p}{q}$ where p and q are integers with no common factors. Then $\cos x + \cos \alpha x$ will be periodic with period $2\pi q$.

Page 117, problem 8. (a) First suppose that $f(x)$ is an even function, so $f(-x) = f(x)$. Then by the chain rule,

$$\frac{d}{dx} f(-x) = -f'(-x)$$

but differentiating the equation $f(-x) = f(x)$ gives

$$\frac{d}{dx} f(-x) = f'(x)$$

Putting the right sides of these two gives $-f'(-x) = f'(x)$, or $f'(-x) = -f'(x)$, so f' is an odd function if f is even.

Next, suppose that $f(x)$ is an odd function, so $f(-x) = -f(x)$. Once again, by the chain rule,

$$\frac{d}{dx} f(-x) = -f'(-x)$$

and differentiating the equation $f(-x) = -f(x)$ gives

$$\frac{d}{dx} f(-x) = -f'(x)$$

Putting the right sides of these two gives $-f'(-x) = -f'(x)$, or $f'(-x) = f'(x)$, so f' is an even function if f is odd.

(b) One way to "ignore" the constant of integration is to do definite integration starting at $x = 0$, so we write

$$F(x) = \int_0^x f(t)\, dt.$$

Then

$$F(-x) = \int_0^{-x} f(t)\, dt$$

Suppose $f(x)$ is even, so that $f(-x) = f(x)$. Then let $q = -t$ so that

$$F(-x) = -\int_0^x f(-q)\, dq = -\int_0^x f(q)\, dq = -F(x)$$

showing that $F(x)$ is odd.

Likewise, if $f(x)$ is odd, so that $f(-x) = -f(x)$, then:

$$F(-x) = \int_0^{-x} f(t)\, dt = -\int_0^x f(-q)\, dq = \int_0^x f(q)\, dq = F(x)$$

showing that $F(x)$ is even.

Page 122, problem 3. Since $u_{tt} = c^2 u_{xx}$ and $u(0,t) = 0$ and $u_x(\ell, t) = 0$, we know that the eigenvalue problem for $X(x)$ will be

$$X'' + \lambda X = 0 \qquad X(0) = 0 \quad X'(\ell) = 0$$

We know that there are only positive eigenvalues. The condition $X(0) = 0$ together with the differential equation tells us that $X(x) = c\sin(\sqrt{\lambda}\, x)$, and then $X'(\ell) = c\sqrt{\lambda}\cos(\sqrt{\lambda}\,\ell)$, which is zero (with non-zero c) provided

$$\lambda = \frac{(2k+1)^2\pi^2}{4\ell^2} \qquad \text{for} \quad k = 0, 1, 2, \ldots$$

giving us the eigenfunctions

$$X_k(x) = \sin\left(\frac{(2k+1)\pi x}{2\ell}\right) \qquad \text{for} \quad k = 0, 1, 2, \ldots$$

The corresponding solutions for $T(t)$ are

$$T_k(t) = A_k \cos\left(\frac{(2k+1)\pi ct}{2\ell}\right) + B_k \sin\left(\frac{(2k+1)\pi ct}{2\ell}\right)$$

so that, so far,

$$u(x,t) = \sum_{k=0}^{\infty} \sin\left(\frac{(2k+1)\pi x}{2\ell}\right)\left[A_k \cos\left(\frac{(2k+1)\pi ct}{2\ell}\right) + B_k \sin\left(\frac{(2k+1)\pi ct}{2\ell}\right)\right]$$

The condition $u_t(x,0) = 0$ tells us that all the B_k are zero, and the condition $u(x,0) = x$ tells us that we need to choose the A_k so that

$$x \sim \sum_{k=0}^{\infty} A_k \sin\left(\frac{(2k+1)\pi x}{2\ell}\right)$$

This means that

$$A_k = \frac{\displaystyle\int_0^\ell x \sin\left(\frac{(2k+1)\pi x}{2\ell}\right)\,dx}{\displaystyle\int_0^\ell \sin^2\left(\frac{(2k+1)\pi x}{2\ell}\right)\,dx}$$

Integrating the numerator by parts gives:

$$\begin{aligned}\int_0^\ell x \sin\left(\frac{(2k+1)\pi x}{2\ell}\right)\,dx &= -\frac{2\ell x}{(2k+1)\pi}\cos\left(\frac{(2k+1)\pi x}{2\ell}\right) + \frac{4\ell^2}{(2k+1)^2\pi^2}\sin\left(\frac{(2k+1)\pi x}{2\ell}\right)\Bigg|_0^\ell \\ &= \frac{(-1)^k 4\ell^2}{(2k+1)^2\pi^2}\end{aligned}$$

And the standard sine-squared identity trick shows that the value of the integral in the denominator is $\frac{1}{2}\ell$. Therefore

$$A_k = \frac{\dfrac{(-1)^k 4\ell^2}{(2k+1)^2\pi^2}}{\dfrac{\ell}{2}} = \frac{(-1)^k 8\ell}{(2k+1)^2\pi^2}$$

and the solution of the problem is

$$u(x,t) = \sum_{k=0}^{\infty} \frac{(-1)^k 8\ell}{(2k+1)^2\pi^2} \sin\left(\frac{(2k+1)\pi x}{2\ell}\right)\cos\left(\frac{(2k+1)\pi ct}{2\ell}\right)$$

Page 122, problem 5. (a) We're assuming that the eigenvalue problem is for the differential equation $X'' + \lambda X = 0$, of course. Then the boundary conditions $u(0,t) = 0$ and $u_x(\ell,t) = 0$ become $X(0) = 0$ and $X'(\ell) = 0$. The solutions of $X'' + \lambda X = 0$ (for $\lambda > 0$) are

$$X(x) = c_1 \sin\sqrt{\lambda}x + c_2 \cos\sqrt{\lambda}x$$

Setting $x = 0$ gives $X(0) = c_2$, which tells us that $c_2 = 0$. Now we have $X'(x) = \sqrt{\lambda}c_1 \cos\sqrt{\lambda}x$, and setting $x = \ell$ gives $X'(\ell) = \sqrt{\lambda}c_1 \cos\sqrt{\lambda}\ell$. If this is zero then $\sqrt{\lambda}\ell$ must be an odd multiple of $\pi/2$, so

$$\lambda_n = \frac{(2n+1)^2\pi^2}{4\ell^2} \qquad \text{and} \qquad X_n = c_n \sin\left(\frac{(2n+1)\pi x}{2\ell}\right).$$

(b) Let $\overline{\varphi}(x)$ be defined as in the problem to be the even extension of φ from the interval $[0,\ell]$ to the interval $[0,2\ell]$. Then $\overline{\varphi}$ is zero at both endpoints, $x = 0$ and $x = 2\ell$ so it's appropriate to write its Fourier sine series:

$$\overline{\varphi}(x) \sim \sum_{k=1}^{\infty} C_k \sin\left(\frac{k\pi x}{2\ell}\right) \quad \text{where} \quad C_k = \frac{2}{2\ell}\int_0^{2\ell} \overline{\varphi}(x)\sin\left(\frac{k\pi x}{2\ell}\right)\,dx.$$

(c) In the integral for C_k, make the change of variables $y = 2\ell - x$. Then $x = 2\ell - y$ and $dx = -dy$ and the integral becomes:

$$C_k = \frac{1}{\ell}\int_{2\ell}^{0} \overline{\varphi}(2\ell - y)\sin\left(\frac{k\pi(2\ell - y)}{2\ell}\right)(-dy) = \frac{1}{\ell}\int_0^{2\ell} \overline{\varphi}(2\ell - y)\sin\left(\frac{k\pi(2\ell - y)}{2\ell}\right)\,dy$$

Now $\overline{\varphi}(2\ell - y) = \overline{\varphi}(y)$ on the interval $x \in [0, 2\ell]$ by the definition of $\overline{\varphi}$, and if k is *even*, than

$$\sin\left(\frac{k\pi(2\ell - y)}{2\ell}\right) = \sin\left(2k\pi - \frac{k\pi y}{2\ell}\right) = \sin\left(-\frac{k\pi y}{2\ell}\right) = -\sin\left(\frac{k\pi y}{2\ell}\right)$$

by the 2π-periodicity and oddness of the sine function. Continuing the chain of equalities for C_k, we now have, for k even,

$$C_k = \cdots = -\frac{1}{\ell}\int_0^{2\ell} \overline{\varphi}(y)\sin\left(\frac{k\pi y}{2\ell}\right)\,dy = -C_k$$

which shows that $C_k = 0$ for k even.

(d) But for odd k, so we'll write $k = 2m + 1$, we have that

$$\sin\left(\frac{(2m+1)\pi(2\ell - y)}{2\ell}\right) = \sin\left(\frac{(2m+1)\pi y}{2\ell}\right)$$

so the part of the integral that defines C_{2m+1} on the interval $[\ell, 2\ell]$ gives the same value as the part on $[0, \ell]$, as we will show. We can write (making the change of variable $y = 2\ell - x$ in the second integral and using the symmetry of $\overline{\varphi}$ and the sine function again):

$$\begin{aligned}
C_{2m+1} &= \frac{1}{\ell}\int_0^{\ell} \varphi(x)\sin\left(\frac{(2m+1)\pi x}{2\ell}\right)dx + \frac{1}{\ell}\int_{\ell}^{2\ell} \overline{\varphi}(x)\sin\left(\frac{(2m+1)\pi x}{2\ell}\right)dx \\
&= \frac{1}{\ell}\int_0^{\ell} \varphi(x)\sin\left(\frac{(2m+1)\pi x}{2\ell}\right)dx - \frac{1}{\ell}\int_{\ell}^{0} \overline{\varphi}(2\ell - y)\sin\left(\frac{(2m+1)\pi(2\ell - y)}{2\ell}\right)dy \\
&= \frac{1}{\ell}\int_0^{\ell} \varphi(x)\sin\left(\frac{(2m+1)\pi x}{2\ell}\right)dx + \frac{1}{\ell}\int_0^{\ell} \overline{\varphi}(y)\sin\left(\frac{(2m+1)\pi y}{2\ell}\right)dy \\
&= \frac{1}{\ell}\int_0^{\ell} \varphi(x)\sin\left(\frac{(2m+1)\pi x}{2\ell}\right)dx + \frac{1}{\ell}\int_0^{\ell} \varphi(y)\sin\left(\frac{(2m+1)\pi y}{2\ell}\right)dy \\
&= \frac{2}{\ell}\int_0^{\ell} \varphi(x)\sin\left(\frac{(2m+1)\pi x}{2\ell}\right)dx
\end{aligned}$$

So, letting $k = 2m + 1$,

$$C_k = \frac{2}{\ell}\int_0^{\ell} \varphi(x)\sin\left(\frac{(2m+1)\pi x}{2\ell}\right)dx$$

This is the same formula we found and used (for A_k) in problem 3 above.

Page 122, problem 12. Integrate by parts with $u = g(x)$ and $dv = f''(x)\,dx$, so that $du = g'(x)\,dx$ and $v = f'(x)$, and get

$$\int_a^b f''(x)g(x)\,dx = f'(x)g(x)\Big|_a^b - \int_a^b f'(x)g'(x)\,dx$$

which is Green's first identity.

Write up solutions of the following to hand in:

- Page 100 (page 97 in the first ed), problems 7, 11, 14, 17
- Page 111 (page 107 in the first ed), problems 5, 6, 7, 10, 11
- Page 117 (page 113 in the first ed), problems 11, 14
- Page 122 (page 118 in the first ed), problems 2, 4, 13

Page 100, problem 7. We're considering the eigenvalues of $X'' + \lambda X = 0$ with $X'(0) - aX(0) = 0$ and $X'(\ell) + aX(\ell) = 0$ for large positive values of a. As explained in the text, there are only positive eigenvalues for $a > 0$ which satisfy $\lambda = \beta^2$ for numbers β that satisfy

$$\tan \beta\ell = \frac{2a\beta}{\beta^2 - a^2}.$$

As $a \to \infty$, the right side of this equation approaches zero, so the values of β that satisfy the equation approach numbers that satisfy $\beta\ell = n\pi$, in other words $\beta = \dfrac{n\pi}{\ell}$

Page 100, problem 11. Write

$$E(t) = \frac{1}{2c^2} \int_0^\ell u_t^2 + c^2 u_x^2 \, dx.$$

We'll show that E is a constant by showing that its derivative is zero.

$$\frac{dE}{dt} = \frac{1}{c^2} \int_0^\ell u_t u_{tt} + c^2 u_x u_{xt} \, dx.$$

Now integrate the second term by parts with $f = u_x$ and $dg = u_{xt}\, dx$ so that $df = u_{xx}\, dx$ and $g = u_t$. The result is

$$\frac{dE}{dt} = \frac{1}{c^2} \int_0^\ell u_t u_{tt} - c^2 u_t u_{xx} \, dx + u_x u_t \Big|_0^\ell$$

Now the integral is zero because the integrand is $u_t(u_{tt} - c^2 u_{xx})$, which is zero because u satisfies the wave equation. So

$$\frac{dE}{dt} = u_x(\ell, t)u_t(\ell, t) - u_x(0, t)u_t(0, t).$$

(a) Since u is constantly zero for $x = 0$ and $x = \ell$, we have $u_t(0, t) = u_t(\ell, t) = 0$, therefore $dE/dt = 0$ and E is constant in this case.

(b) For Neumann conditions we have that $u_x(0, t) = u_x(\ell, t) = 0$ so $dE/dt = 0$ again and E is constant.

(c) We have that $E_R = E + \frac{1}{2}a_\ell(u(\ell, t))^2 + \frac{1}{2}a_0(u(0, t))^2$ (to account for the energy being radiated or absorbed at the ends of the interval) and so

$$\begin{aligned}
\frac{dE_R}{dt} &= [u_x(\ell, t)u_t(\ell, t) - u_x(0, t)u_t(0, t)] + a_\ell u(\ell, t)u_t(\ell, t) + a_0 u(0, t)u_t(0, t) \\
&= u_t(\ell, t)[u_x(\ell, t) + a_\ell u(\ell, t)] - u_t(0, t)[u_x(0, t) - a_0 u(0, t)] \\
&= 0
\end{aligned}$$

since the expressions in brackets are precisely the Robin boundary conditions.

Page 100, problem 14. The problem is to find eigenvalues/eigenfunctions for

$$x^2u'' + 3xu' + \lambda u = 0 \quad \text{on } 1 < x < e \quad \text{with } u(1) = u(e) = 0$$

Following the hint (although a better hint would have been to let $x = e^t$), we'll assume that $u = x^m$. Putting this into the differential equation gives:

$$x^2 m(m-1)x^{m-2} + 3xmx^{m-1} + \lambda x^m = 0$$

$$x^m[m(m-1) + 3m + \lambda] = 0$$

$$x^m[m^2 + 2m + \lambda] = 0$$

For this to be true for all x we must choose m so that the quadratic is zero, so

$$m = -1 \pm \tfrac{1}{2}\sqrt{4 - 4\lambda} = -1 \pm i\sqrt{\lambda - 1}.$$

With these values of m, we get

$$u = c_1 x^{-1+i\sqrt{\lambda-1}} + c_2 x^{-1-i\sqrt{\lambda-1}} = c_1 \frac{e^{i\sqrt{\lambda-1}\,\ln x}}{x} + c_2 \frac{e^{-i\sqrt{\lambda-1}\,\ln x}}{x} = a_1 \frac{\cos(\sqrt{\lambda-1}\,\ln x)}{x} + a_2 \frac{\sin\sqrt{\lambda-1}\,\ln x}{x}$$

after renaming the constants. Now $u(1) = 0$ implies that $a_1 = 0$, and $u(e) = 0$ implies that $\sin\sqrt{\lambda - 1} = 0$, or that $\sqrt{\lambda - 1} = n\pi$ or

$$\lambda_n = 1 + n^2\pi^2 \quad \text{for } n = 1, 2, \ldots$$

and

$$u_n(x) = \frac{\sin(n\pi \ln x)}{x}.$$

Page 100, problem 17. We're looking for *positive* eigenvalues and their eigenfunctions for $X'''' = \lambda X$ in $0 < x < \ell$ with $X(0) = X(\ell) = X'(0) = X'(\ell) = 0$. Since $\lambda > 0$, we'll let $\lambda = \beta^4$. Then the roots of $r^4 = \lambda r$ are β, $-\beta$, $i\beta$ and $-i\beta$. So the general solution of the equation is

$$X = c_1 \cosh\beta x + c_2 \sinh\beta x + c_3 \cos\beta x + c_4 \sin\beta x,$$

which has derivative

$$X' = \beta c_1 \sinh\beta x + \beta c_2 \cosh\beta x - \beta c_3 \sin\beta x + \beta c_4 \cos\beta x.$$

Now $X(0) = 0$ tells us that $c_1 + c_3 = 0$, so $c_3 = -c_1$, and $X'(0) = 0$ tells us that $\beta c_2 + \beta c_4 = 0$, so $c_4 = -c_2$. So at this point we can write the solution as

$$X = c_1(\cosh\beta x - \cos\beta x) + c_2(\sinh\beta x - \sin\beta x)$$

which has derivative

$$X' = \beta c_1(\sinh\beta x + \sin\beta x) + \beta c_2(\cosh\beta x - \cos\beta x).$$

Now we need to use the conditions $X(\ell) = X'(\ell) = 0$, which are

$$c_1(\cosh\beta\ell - \cos\beta\ell) + c_2(\sinh\beta\ell - \sin\beta\ell) = 0$$

$$\beta c_1(\sinh\beta\ell + \sin\beta\ell) + \beta c_2(\cosh\beta\ell - \cos\beta\ell) = 0$$

This is two linear homogeneous equations in the two unknowns c_1 and c_2, so we need its determinant to be zero. In other words, we need β to satisfy (after dividing out β from the second equation)

$$(\cosh\beta\ell - \cos\beta\ell)^2 - (\sinh\beta\ell + \sin\beta\ell)(\sinh\beta\ell - \sin\beta\ell) = 0$$

or

$$\cosh^2\beta\ell - 2\cos\beta\ell\cosh\beta\ell + \cos^2\beta\ell - \sinh^2\beta\ell + \sin^2\beta\ell = 0$$

or (using the trig identity $\cos^2\theta + \sin^2\theta = 1$ and the hyperbolic identity $\cosh^2 s - \sinh^2 s = 1$)

$$2 - 2\cos\beta\ell\cosh\beta\ell = 0 \quad \text{in other words} \quad \cos\beta\ell = \frac{1}{\cosh\beta\ell}.$$

Now for even moderately large x, $\cosh x \approx \frac{1}{2}e^x$, so the right side of the equation on the right is small, positive and approaching zero — this means that its solutions are close to (and getting closer to as β increases) odd multiples of $\pi/(2\ell)$, and there is an infinite sequence of them.

If β is a root of $\cos\beta\ell\cosh\beta\ell = 1$, tenth two equations $X(\ell) = 0$ and $X'(\ell) = 0$ are *redundant*, so we solve the first one and get

$$c_2 = -\frac{\cosh\beta\ell - \cos\beta\ell}{\sinh\beta\ell - \sin\beta\ell}\, c_1 \quad (\text{and } c_3 = -c1,\ c_4 = -c2)$$

If we set $c_1 = \sinh\beta\ell - \sin\beta\ell$ to get rid of the fraction, then the eigenfunction is

$$X(x) = (\sinh\beta\ell - \sin\beta\ell)(\cosh\beta x - \cos\beta x) - (\cosh\beta\ell - \cos\beta\ell)(\sinh\beta x - \sin\beta x)$$

where

$$\beta \text{ satisfies} \quad \cos\beta\ell = \frac{1}{\cosh\beta\ell} \quad \text{and } \lambda = \beta^4$$

Page 111, problem 5. The Fourier sine series of $\varphi(x) = x$ on $(0, \ell)$ is

$$\sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{\ell}$$

where

$$A_n = \frac{2}{\ell}\int_0^\ell x\sin\frac{n\pi x}{\ell}\,dx = \frac{2}{\ell}\left[-\frac{\ell x}{n\pi}\cos\frac{n\pi x}{\ell} + \frac{ell^2}{n^2\pi^2}\sin\frac{n\pi x}{\ell}\right]_0^\ell = -\frac{2\ell}{n\pi}\cos n\pi = (-1)^{n+1}\frac{2\ell}{n\pi}.$$

So

$$x \sim \sum_{n=1}^{\infty}(-1)^{n+1}\frac{2\ell}{n\pi}\sin\frac{n\pi x}{\ell}.$$

The series converges to the *odd* 2ℓ-periodic extension of $\varphi(x)$.

(a) Now

$$\frac{x^2}{2} = \int_0^x t\,dt \sim \sum_{n=1}^{\infty}(-1)^{n+1}\frac{2\ell}{n\pi}\int_0^x \sin\frac{n\pi t}{\ell}\,dt = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{2\ell^2}{n^2\pi^2}\left[-\cos\frac{n\pi t}{\ell}\right]_0^x$$

$$= \sum_{n=1}^{\infty}(-1)^{n}\frac{2\ell^2}{n^2\pi^2}\cos\frac{n\pi x}{\ell} + \sum_{n=1}^{\infty}(-1)^{n+1}\frac{2\ell^2}{n^2\pi^2}$$

where now the series represents the *even* 2ℓ-periodic extension of $\frac{1}{2}x^2$. To evaluate the second (numerical) series in the last line, integrate both sides from $-\ell$ to ℓ. Then all the $\cos(n\pi x/\ell)$ terms will integrate to zero, being integrated over a whole number of periods, and we get

$$\int_{-\ell}^{\ell} \frac{x^2}{2}\,dx = \frac{\ell^3}{3} = 2\ell \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2\ell^2}{n^2\pi^2}$$

which tells us that

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{2\ell^2}{n^2\pi^2} = \frac{\ell^2}{6}$$

Thus

$$\frac{x^2}{2} \sim \frac{\ell^2}{6} + \sum_{n=1}^{\infty} (-1)^n \frac{2\ell^2}{n^2\pi^2} \cos\frac{n\pi x}{\ell}.$$

(b) At $x = 0$, this series gives

$$0 = \frac{\ell^2}{6} + \sum_{n=1}^{\infty} (-1)^n \frac{2\ell^2}{n^2\pi^2}$$

which implies (adding the $\ell^2/6$ to both sides and then multiplying both sides by $-\pi^2/(2\ell^2)$) that

$$\frac{\pi^2}{12} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}.$$

Page 111, problem 6. In problem 5(a), we calculated the Fourier cosine series for $\frac{1}{2}x^2$. Multiply this series by 6 to obtain:

$$3x^2 \sim \ell^2 + \sum_{n=1}^{\infty} (-1)^n \frac{12\ell^2}{n^2\pi^2} \cos\frac{n\pi x}{\ell}.$$

Then

$$x^3 = \int_0^x 3t^2\,dt \sim \int_0^x \ell^2\,dt + \frac{12\ell^2}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \int_0^x \cos\frac{n\pi t}{\ell}\,dt = \ell^2 x + \frac{12\ell^3}{\pi^3} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} \sin\frac{n\pi x}{\ell}$$

But we know the sine series for x from the beginning of problem 5, so we get

$$x^3 \sim \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 2\ell^3}{n\pi} \sin\frac{n\pi x}{\ell} + \frac{12\ell^3}{\pi^3} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} \sin\frac{n\pi x}{\ell} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 2\ell^3}{\pi} \left(\frac{1}{n} - \frac{6}{\pi^2 n^3}\right) \sin\frac{n\pi x}{\ell}.$$

This is the Fourier sine series for x^3.

(b) Starting from the series we just derived:

$$4x^3 \sim \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 8\ell^3}{\pi} \left(\frac{1}{n} - \frac{6}{\pi^2 n^3}\right) \sin\frac{n\pi x}{\ell}$$

Therefore

$$x^4 = \int_0^x 4t^3\,dt \sim \sum_{n=1}^{\infty} \frac{(-1)^{n+1}8\ell^3}{\pi}\left(\frac{1}{n} - \frac{6}{\pi^2 n^3}\right)\int_0^x \sin\frac{n\pi t}{\ell}\,dt$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n 8\ell^4}{n\pi^2}\left(\frac{1}{n} - \frac{6}{\pi^2 n^3}\right)\cos\frac{n\pi t}{\ell}\Big|_0^x$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n 8\ell^4}{n\pi^2}\left(\frac{1}{n} - \frac{6}{\pi^2 n^3}\right)\cos\frac{n\pi x}{\ell} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}8\ell^4}{\pi^2}\left(\frac{1}{n^2} - \frac{6}{\pi^2 n^4}\right)$$

Now, as in problem 5(a), integrate both sides from $-\ell$ to ℓ and learn that

$$\int_{-\ell}^{\ell} x^4\,dx = \frac{2\ell^5}{5} = 2\ell\sum_{n=1}^{\infty} \frac{(-1)^{n+1}8\ell^4}{\pi^2}\left(\frac{1}{n^2} - \frac{6}{\pi^2 n^4}\right)$$

so that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}8\ell^4}{\pi^2}\left(\frac{1}{n^2} - \frac{6}{\pi^2 n^4}\right) = \frac{\ell^4}{5}$$

from which we can conclude that

$$x^4 \sim \frac{\ell^4}{5} + \sum_{n=1}^{\infty} \frac{(-1)^n 8\ell^4}{n\pi^2}\left(\frac{1}{n} - \frac{6}{\pi^2 n^3}\right)\cos\frac{n\pi x}{\ell}$$

Page 111, problem 7. Set $x = 0$ in the last equation of 6(b) to get

$$0 = \frac{\ell^4}{5} + \sum_{n=1}^{\infty} \frac{(-1)^n 8\ell^4}{n\pi^2}\left(\frac{1}{n} - \frac{6}{\pi^2 n^3}\right)$$

which tells us that

$$\frac{\ell^4}{5} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}8\ell^4}{\pi^2 n^2} - \sum_{n=1}^{\infty} \frac{(-1)^{n+1}48\ell^4}{\pi^4 n^4}$$

Now from 5(b) we know that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$$

so we can conclude that

$$\frac{\ell^4}{5} = \frac{2\ell^4}{3} - \sum_{n=1}^{\infty} \frac{(-1)^{n+1}48\ell^4}{\pi^4 n^4}$$

Divide everything by ℓ^4 and see that

$$\frac{48}{\pi^4}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4} = \frac{2}{3} - \frac{1}{5} = \frac{7}{15}$$

from which we can conclude that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4} = \frac{7\pi^4}{15\cdot 48} = \frac{7\pi^4}{720}.$$

Page 111, problem 10. We need to solve the wave equation $u_{tt} = c^2 u_{xx}$ where $c = \frac{T}{\rho}$, with boundary data $u(0,t) = u(\ell,t) = 0$ and initial data

$$u(x,0) = 0 \qquad u_t(x,0) = \psi(x) = \begin{cases} V & \text{for } x \in [\frac{1}{2}L - \delta\,,\ \frac{1}{2}L + \delta] \\ 0 & \text{otherwise} \end{cases}$$

We know that the solution is

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{\ell} \sin \frac{cn\pi t}{\ell}$$

where $\frac{cn\pi}{L} A_n \sin \frac{n\pi x}{\ell}$ is the Fourier sine series for $\psi(x)$, which implies

$$\frac{cn\pi}{\ell} A_n = \frac{2}{\ell} \int_{\frac{1}{2}\ell-\delta}^{\frac{1}{2}\ell+\delta} V \sin \frac{n\pi x}{\ell}\, dx = -\frac{2V}{n\pi} \cos \frac{n\pi x}{\ell}\bigg|_{\frac{1}{2}\ell-\delta}^{\frac{1}{2}\ell+\delta} = \begin{cases} 0 & \text{if } n \text{ is even} \\ \dfrac{(-1)^k 4V}{(2k-1)\pi} \sin \dfrac{(2k-1)\pi\delta}{\ell} & \text{if } n = 2k-1 \text{ is odd} \end{cases}$$

To get to the last expression, note that

$$\begin{aligned} \cos \frac{n\pi x}{\ell}\bigg|_{\frac{1}{2}\ell-\delta}^{\frac{1}{2}\ell+\delta} &= \cos\left(\frac{n\pi}{2} + \frac{n\pi\delta}{\ell}\right) - \cos\left(\frac{n\pi}{2} - \frac{n\pi\delta}{\ell}\right) \\ &= \left[\cos\frac{n\pi}{2}\cos\frac{n\pi\delta}{\ell} - \sin\frac{n\pi}{2}\sin\frac{n\pi\delta}{\ell}\right] - \left[\cos\frac{n\pi}{2}\cos\frac{n\pi\delta}{\ell} + \sin\frac{n\pi}{2}\sin\frac{n\pi\delta}{\ell}\right] \\ &= -2\sin\frac{n\pi}{2}\sin\frac{n\pi\delta}{\ell} \end{aligned}$$

If n is even, then $n\pi/2$ is an integer multiple of π so that $\sin n\pi/2 = 0$. If $n = 2k-1$ is odd, then $\sin(k - \frac{1}{2})\pi = (-1)^{k+1}$.

Therefore

$$u(x,t) = \sum_{k=1}^{\infty} \frac{(-1)^k 4\ell V}{c(2k-1)^2\pi^2} \sin\frac{(2k-1)\pi\delta}{\ell} \sin\frac{(2k-1)\pi x}{\ell} \sin\frac{c(2k-1)\pi t}{\ell}.$$

The kth harmonic h_k is the kth term of this series, and its energy is (writing n instead of $2k-1$ to save space):

$$\begin{aligned} E(h_k) &= \frac{1}{2}\int_0^{\ell} \left[\rho\left(\frac{\partial h_k}{\partial t}\right)^2 + T\left(\frac{\partial h}{\partial x}\right)^2\right] dx \\ &= \frac{8\ell^2 V^2}{c^2 n^4 \pi^4} \sin^2\frac{n\pi\delta}{\ell} \int_0^{\ell} \rho\frac{c^2 n^2\pi^2}{\ell^2} \sin^2\frac{n\pi x}{\ell}\cos^2\frac{cn\pi t}{\ell} + T\frac{n^2\pi^2}{\ell^2}\cos^2\frac{n\pi x}{\ell}\sin^2\frac{cn\pi t}{\ell}\, dx \end{aligned}$$

Now the integrals of both $sin^2(n\pi x/\ell)$ and $\cos^2(n\pi x/\ell)$ from 0 to ℓ equal $\frac{1}{2}\ell$, by the usual double-angle formula trick. We can also use the fact that $T = c^2\rho$ to arrive at

$$E(h_k) = \frac{8V^2\rho}{n^2\pi^2}\sin^2\frac{n\pi\delta}{\ell}\left[\frac{\ell}{2}\cos^2\frac{c^2 n\pi t}{\ell} + \sin^2\frac{c^2 n\pi t}{\ell}\right] = \frac{4V^2\rho\ell}{n^2\pi^2}\sin^2\frac{n\pi\delta}{\ell}$$

Now, if $n\delta$ is small (compared to ℓ), then we can use the small-θ approximation $\sin\theta \approx \theta$ to conclude:

$$E(h_k) \approx \frac{4V^2\rho\ell}{n^2\pi^2} \cdot \frac{n^2\pi^2\delta^2}{\ell^2} = \frac{4V^2\rho\delta^2}{\ell}$$

which is indeed independent of n (or k), so the energy of the first few (odd) overtones is about the same (and slowly decreasing, since that's what the sine does).

Page 111, problem 11. Just as in problem 10 above, the solution of the initial-boundary problem for the wave equation is

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{\ell} \sin\frac{cn\pi t}{\ell}$$

where this time, $\dfrac{cn\pi}{L} A_n \sin\dfrac{n\pi x}{\ell}$ is the Fourier sine series for

$$\psi(x) = \begin{cases} V & \text{for } x \in [a-\delta\,,\, a+\delta] \\ 0 & \text{otherwise} \end{cases}$$

where a is a node (zero) of $\sin\dfrac{n\pi x}{\ell}$, in other words a is one of

$$a \in \left\{\frac{\ell}{n}, \frac{2\ell}{n}, \cdots, \frac{k\ell}{n}, \cdots, \frac{(n-1)\ell}{n}\right\}.$$

We want to show if this is the case that $A_n = 0$. We know that A_n is a multiple of

$$\begin{aligned}
\int_0^\ell \psi(x)\sin\frac{n\pi x}{\ell}\,dx = V\int_{\frac{k\ell}{n}-\delta}^{\frac{k\ell}{n}+\delta} \sin\frac{n\pi x}{\ell}\,dx &= -\frac{\ell V}{n\pi}\cos\frac{n\pi x}{\ell}\Bigg|_{\frac{k\ell}{n}-\delta}^{\frac{k\ell}{n}+\delta} \\
&= \frac{\ell V}{n\pi}\left[\cos\left(k\pi - \frac{n\pi\delta}{\ell}\right) - \cos\left(k\pi + \frac{n\pi\delta}{\ell}\right)\right] \\
&= \frac{\ell V}{n\pi}\left[\left(\cos k\pi\cos\frac{n\pi\delta}{\ell} + \sin k\pi\sin\frac{n\pi\delta}{\ell}\right) - \left(\cos k\pi\cos\frac{n\pi\delta}{\ell} - \sin k\pi\sin\frac{n\pi\delta}{\ell}\right)\right] \\
&= \frac{2\ell V}{n\pi}\sin k\pi\sin\frac{n\pi\delta}{\ell} = 0
\end{aligned}$$

because $\sin k\pi = 0$.

Page 117, problem 11. For the Fourier series of e^x, the coefficients of the complex form

$$e^x \sim \sum_{n=-\infty}^{\infty} A_n e^{in\pi x/\ell}$$

are

$$\begin{aligned}
A_n &= \frac{1}{2\ell}\int_{-\ell}^{\ell} e^x e^{-in\pi x/\ell}\,dx = \frac{1}{2\ell}\int_{-\ell}^{\ell} e^{\left(1-\frac{in\pi}{\ell}\right)x}\,dx \\
&= \frac{1}{2\ell\left(1-\frac{in\pi}{\ell}\right)}\left[e^{\left(1-\frac{in\pi}{\ell}\right)\ell} - e^{-\left(1-\frac{in\pi}{\ell}\right)\ell}\right] = \frac{1}{2(\ell - in\pi)}\left[e^\ell e^{-in\pi} - e^{-\ell}e^{in\pi}\right] \\
&= \frac{\ell + in\pi}{\ell^2+n^2\pi^2}(-1)^n\frac{e^\ell - e^{-\ell}}{2} = (-1)^n\frac{\ell+in\pi}{\ell^2+n^2\pi^2}\sinh\ell
\end{aligned}$$

Therefore

$$e^x \sim \sum_{-\infty}^{\infty} (-1)^n \sinh \ell \, \frac{\ell + in\pi}{\ell^2 + n^2\pi^2} \, e^{in\pi x/\ell}.$$

For the real form, first notice that the $n = 0$ term is $\dfrac{\sinh \ell}{\ell}$. Then put the nth term together with the $-n$th term and get:

$$\begin{aligned} e^x &\sim \frac{\sinh \ell}{\ell} + \sum_{n=1}^{\infty} (-1)^n \frac{\sinh \ell}{\ell^2 + n^2\pi^2} \left[(\ell + in\pi) e^{in\pi x/\ell} + (\ell - in\pi) e^{-in\pi x} \ell \right] \\ &= \frac{\sinh \ell}{\ell} + \sum_{n=1}^{\infty} \frac{(-1)^n \sinh \ell}{\ell^2 + n^2\pi^2} \left(2\ell \cos \frac{n\pi x}{\ell} - 2n\pi \sin \frac{n\pi x}{\ell} \right). \end{aligned}$$

Page 117, problem 14. This time we want the Fourier series for $|x|$,. For its complex form

$$|x| \sim \sum_{n=-\infty}^{\infty} A_n e^{in\pi x/\ell}$$

the coefficients are

$$A_n = \frac{1}{2\ell} \int_{-\ell}^{\ell} |x| e^{-in\pi x/\ell} \, dx = \frac{1}{2\ell} \left[\int_0^{\ell} x e^{-in\pi x/\ell} \, dx - \int_{-\ell}^{0} x e^{-in\pi x/\ell} \, dx \right]$$

Make the substitution $s = -x$, so $ds = -dx$ in the second integral, and then just change the s back to x and get

$$A_n = \frac{1}{2\ell} \int_0^{\ell} x \left[e^{-in\pi x/\ell} + e^{in\pi x/\ell} \right] dx = \frac{1}{\ell} \int_0^{\ell} x \cos \frac{n\pi x}{\ell} \, dx$$

Now integrate by parts with $u = x$ and $dv = \cos \dfrac{n\pi x}{\ell} \, dx$ and get

$$A_n = \frac{1}{\ell} \left[\frac{x\ell}{n\pi} \sin \frac{n\pi x}{\ell} + \frac{\ell^2}{n^2\pi^2} \cos \frac{n\pi x}{\ell} \right]_0^{\ell} = \frac{\ell}{n^2\pi^2} \Big((-1)^n - 1 \Big) = \begin{cases} -\dfrac{2\ell}{n^2\pi^2} & \text{for } n \text{ odd} \\ 0 & \text{for } n \text{ even} \end{cases}$$

As usual, the $n = 0$ term is exceptional:

$$A_0 = \frac{1}{2\ell} \int_{-\ell}^{\ell} |x| \, dx = \frac{\ell}{2}.$$

So we conclude that

$$|x| \sim \frac{\ell}{2} - \sum_{\substack{n \text{ odd} \\ n=-\infty}}^{\infty} \frac{2\ell}{n^2\pi^2} e^{in\pi x/\ell}$$

This is the complex form. Put the positive and negative n terms together to make cosines for the real form:

$$|x| \sim \frac{\ell}{2} - \sum_{\substack{n \text{ odd} \\ n=1}}^{\infty} \frac{2\ell}{n^2\pi^2} \left(e^{in\pi x/\ell} + e^{-in\pi x/\ell} \right) = \frac{\ell}{2} - \sum_{\substack{n \text{ odd} \\ n=1}}^{\infty} \frac{4\ell}{n^2\pi^2} \cos \frac{n\pi x}{\ell}$$

Page 122, problem 2. Write $\langle f,g\rangle$ for $\displaystyle\int_{-1}^{1} f(x)g(x)\,dx$. Then

(a) $\langle c,x\rangle = \displaystyle\int_{-1}^{1} cx\,dx = \frac{cx^2}{2}\bigg|_{-1}^{1} = 0$ so any constant c is orthogonal to x.

(b) We want a quadratic polynomial orthogonal to 1 and x, so assume $p(x) = x^2 + bx + c$. Then

$$\langle p,1\rangle = \int_{-1}^{1} x^2 + bx + c\,dx = \frac{1}{3}x^3 + \frac{b}{2}x^2 + cx\bigg|_{-1}^{1} = \frac{2}{3} + 2c$$

If this is zero, then $c = -\frac{1}{3}$. Next,

$$\langle p,x\rangle = \int_{-1}^{1} x^3 + bx^2 + cx\,dx = \frac{1}{4}x^4 + \frac{b}{3}x^3 + \frac{c}{2}x^2\bigg|_{-1}^{1} = \frac{2b}{3}$$

If this is zero, then $b = 0$. So our polynomial is (any constant multiple of) $p(x) = x^2 - \frac{1}{3}$.

(c) For $q(x)$ to be orthogonal to all quadratics, it's enough for it to be orthogonal to 1, x and x^2. If $q(x) = x^3 + bx^2 + cx + d$, then we have (by calculations like those in part (b)):

$$\begin{aligned}\langle q,1\rangle &= \frac{2b}{3} + 2d\\ \langle q,x\rangle &= \frac{2}{5} + \frac{2c}{3}\\ \langle q,x^2\rangle &= \frac{2b}{5} + \frac{2d}{3}\end{aligned}$$

If these three quantities are all zero then the middle one tells us that $c = -\frac{3}{5}$ and the first and last tell us that b and d must be zero. So our polynomial is (any multiple of) $q(x) = x^3 - \frac{3}{5}x$.

Page 122, problem 4. (a) The problem we have to solve is $u_t = ku_{xx}$ with initial condition $u(x,0) = 0$, and boundary conditions $u(0,t) = U$ (a constant) and $u_x(\ell,t) = 0$. We can make the boundary conditions homogeneous by considering the corresponding problem for $v(x,t) = u(x,t) - U$. Then v satisfies

$$v_t = kv_{xx} \quad \text{together with} \quad v(x,0) = -U \quad \text{and} \quad v(0,t) = 0 \quad v_x(\ell,t) = 0.$$

The eigenvalue problem for the separated solutions is

$$X'' + \lambda X = 0 \qquad X(0) = 0 \qquad X'(\ell) = 0.$$

This is a symmetric eigenvalue problem and all the eigenvalues are positive (by Theorem 3), and in fact we've solved this one before:

$$\lambda_n = \frac{(n+\frac{1}{2})^2\pi^2}{\ell^2} \quad \text{and} \quad X_n(x) = \sin\frac{(n+\frac{1}{2})\pi x}{\ell} \quad \text{for} \quad n = 0,1,2,\ldots$$

The corresponding functions of t are

$$T_n(t) = e^{-k(n+\frac{1}{2})^2\pi^2 t/\ell^2}$$

and the solution can be written

$$v(x,t) = \sum_{n=0}^{\infty} A_n e^{-k(n+\frac{1}{2})^2\pi^2 t/\ell^2} \sin\frac{(n+\frac{1}{2})\pi x}{\ell}$$

where

$$\sum_{n=0}^{\infty} A_n \sin\frac{(n+\frac{1}{2})\pi x}{\ell} = -U \quad \text{for } 0 < x < \ell$$

For the coefficients, we know that

$$A_n = \frac{\left\langle -U, \sin\frac{(n+\frac{1}{2})\pi x}{\ell}\right\rangle}{\left\langle \sin\frac{(n+\frac{1}{2})\pi x}{\ell}, \sin\frac{(n+\frac{1}{2})\pi x}{\ell}\right\rangle} = \frac{-\int_0^\ell U\sin\frac{(n+\frac{1}{2})\pi x}{\ell}\,dx}{\int_0^\ell \sin^2\frac{(n+\frac{1}{2})\pi x}{\ell}\,dx}$$

$$= \frac{\frac{U\ell}{(n+\frac{1}{2})\pi}\cos\frac{(n+\frac{1}{2})\pi x}{\ell}\Big|_0^\ell}{\frac{1}{2}\ell} = \frac{-\frac{U\ell}{(n+\frac{1}{2})\pi}}{\frac{1}{2}\ell} = -\frac{2U}{(n+\frac{1}{2})\pi}.$$

Adding back in the U to get u from v, we get:

$$u(x,t) = U - \sum_{n=0}^{\infty} \frac{2U}{(n+\frac{1}{2})\pi} e^{-k(n+\frac{1}{2})^2\pi^2 t/\ell^2} \sin\frac{(n+\frac{1}{2})\pi x}{\ell}$$

(b) To show that the series converges for $t > 0$, begin with the observation that $(n+\frac{1}{2})^2 = n^2 + n + \frac{1}{4} > n$ for all $n \geq 0$. So if $t > 0$ then

$$\frac{k\pi^2 t}{\ell}(n+\frac{1}{2})^2 > \frac{k\pi^2 t}{\ell}\,n$$

which in turn implies that

$$e^{-k(n+\frac{1}{2})^2\pi^2 t/\ell^2} < \left(e^{-k\pi^2 t/\ell^2}\right)^n.$$

And of course we have that $e^{-k\pi^2 t/\ell^2} < 1$. And since $|\sin\theta| \leq 1$ for any value of θ, we have

$$\sum_{n=0}^{\infty}\left|\frac{2U}{(n+\frac{1}{2})\pi} e^{-k(n+\frac{1}{2})^2\pi^2 t/\ell^2} \sin\frac{(n+\frac{1}{2})\pi x}{\ell}\right| < \sum_{n=0}^{\infty}\frac{2U}{(n+\frac{1}{2})\pi}\left(e^{-k\pi^2 t/\ell^2}\right)^n < \sum_{n=0}^{\infty}\frac{2U}{\frac{1}{2}\pi}\left(e^{-k\pi^2 t/\ell^2}\right)^n$$

and the last is a convergent geometric series provided $t > 0$. So the series in $u(x,t)$ converges (absolutely) for all $t > 0$.

(c) At the endpoint $x = \ell$, we have that $\sin\frac{(n+\frac{1}{2})\pi\ell}{\ell} = (-1)^n$ so the series:

$$u(\ell,t) = U - \sum_{n=0}^{\infty}(-1)^n\frac{2U}{(n+\frac{1}{2})\pi} e^{-k(n+\frac{1}{2})^2\pi^2 t/\ell^2}$$

is alternating, the terms are decreasing in size. If we want to see how close $u(\ell,t)$ is to U, we can conclude that the difference is less than the first term of the series. So

$$|u(x,\ell) - U| < \frac{2|U|}{\frac{1}{2}\pi} e^{-\frac{1}{4}k\pi^2 t/\ell^2}.$$

If we want this to be less than ε, then we need

$$e^{-\frac{1}{4}k\pi^2 t/\ell^2} < \frac{\pi\varepsilon}{4|U|}$$

which is equivalent to

$$-\frac{k\pi^2 t}{4\ell^2} < \ln\frac{\pi\varepsilon}{4|U|}$$

or

$$\frac{k\pi^2 t}{4\ell^2} > \ln\frac{4|U|}{\pi\varepsilon}$$

so we need

$$t > \frac{4\ell^2}{k\pi^2}\ln\frac{4|U|}{\pi\varepsilon}.$$

Page 122, problem 13. We need to show that all the eigenvalues of $X'' + \lambda X = 0$ are non-negative if $f(x)f'(x)\Big|_a^b \leq 0$ for all f satisfying the boundary conditions. Suppose $u(x)$ is an eigenfunction with eigenvalue λ. Then

$$\begin{aligned}\lambda\langle u, u\rangle = \langle \lambda u, u\rangle = \langle -u'', u\rangle &= \int_a^b -u''(x)u(x)\,dx \\ &= \int_a^b (u'(x))^2\,dx - u(x)u'(x)\Big|_a^b\end{aligned}$$

by Green's first identity (from problem 12). Since u satisfies the boundary conditions, the last expression is greater than or equal to zero, and we can conclude that $\lambda\langle u, u\rangle \geq 0$. Since $\langle u, u\rangle > 0$, this implies that $\lambda \geq 0$.