Math 425 / AMCS 525 **Dr. DeTurck**
Assignment 7 **Due Tuessday, March 29, 2016**

Topics for this week — Convergence of Fourier series; Laplace's equation and harmonic functions: basic properties, compuations on rectangles and cubes (Fourier!), Poisson's formula for the disk

Seventh Homework Assignment - due Tuesday, March 29

Reading: Read sections 5.4, 5.5, and 6.1 through 6.3 of the text

Be prepared to discuss the following problems in class:

- Page 134 (page 129 in the first ed) problems 12, 13
- Page 145 (page 139 in the first ed) problems 2, 3
- Page 160 (page 154 in the first ed) problems 2, 5, 7
- Page 164 (page 158 in the first ed) problems 1, 6
- *Page 172 (page 163 in the first ed) problems 1, 2

Page 134, problem 12. The Fourier sine series of $f(x) = x$ on $(0, \ell)$ is

$$x \sim \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{\ell}$$

where

$$A_n = \frac{2}{\ell} \int_0^{\ell} x \sin \frac{n\pi x}{\ell}\, dx$$

Integrate by parts with $u = x$ and $dv = \sin \frac{n\pi x}{\ell}\, dx$, so $du = dx$ and $v = -\frac{\ell}{n\pi} \cos \frac{n\pi x}{\ell}$ to get

$$A_n = -\frac{2x}{n\pi} \cos \frac{n\pi x}{\ell}\bigg|_0^{\ell} + \frac{2}{n\pi} \int_0^{\ell} \cos \frac{n\pi x}{\ell}\, dx = \left[-\frac{2x}{n\pi} \cos \frac{n\pi x}{\ell} + \frac{2\ell}{n^2\pi^2} \sin \frac{n\pi x}{\ell} \right]_0^{\ell} = \frac{(-1)^{n+1} 2\ell}{n\pi}$$

Therefore

$$x \sim \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 2\ell}{n\pi} \sin \frac{n\pi x}{\ell}.$$

Parseval's theorem says that

$$\int_0^{\ell} (f(x))^2\, dx = \sum_{n=1}^{\infty} A_n^2 \int_0^{\ell} \sin^2 \frac{n\pi x}{\ell}\, dx$$

so we have

$$\int_0^{\ell} x^2\, dx = \frac{\ell^3}{3} = \sum_{n=1}^{\infty} \frac{4\ell^2}{n^2\pi^2} \cdot \frac{\ell}{2} = \sum_{n=1}^{\infty} \frac{2\ell^3}{n^2\pi^2}$$

Multiply both sides of the equation

$$\frac{\ell^3}{3} = \sum_{n=1}^{\infty} \frac{2\ell^3}{n^2\pi^2} \quad \text{by } \frac{\pi^2}{2\ell^3} \text{ and obtain } \quad \frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Page 134, problem 13. We can obtain the Fourier cosine series for x^2 by integrating the Fourier sine series of x, which we found in the preceding problem:

$$x^2 = \int_0^x 2t\,dt \sim \int_0^x \sum_{n=1}^{\infty} \frac{(-1)^{n+1}4\ell}{n\pi} \sin\frac{n\pi t}{\ell}\,dt = \sum_{n=1}^{\infty} \frac{(-1)^n 4\ell^2}{n^2\pi^2} \cos\frac{n\pi t}{\ell}\bigg|_0^x$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n 4\ell^2}{n^2\pi^2} \cos\frac{n\pi x}{\ell} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}4\ell^2}{n^2\pi^2}$$

We can evaluate the constant sum on the right by integrating both sides from 0 to ℓ (all the cosine terms will integrate to zero) and obtain

$$\int_0^\ell x^2\,dx = \frac{\ell^3}{3} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}4\ell^3}{n^2\pi^2},$$

so, dividing this by ℓ we can conclude that

$$x^2 \sim \frac{\ell^2}{3} + \sum_{n=1}^{\infty} \frac{(-1)^n 4\ell^2}{n^2\pi^2} \cos\frac{n\pi x}{\ell}$$

This time, Parseval's theorem says

$$\int_0^\ell x^4\,dx = \frac{\ell^5}{5} = \frac{\ell^4}{9}\int_0^\ell 1^2\,dx + \sum_{n=1}^{\infty} \frac{16\ell^4}{n^4\pi^4}\int_0^\ell \cos^2\frac{n\pi x}{\ell}\,dx = \frac{\ell^5}{9} + \sum_{n=1}^{\infty} \frac{8\ell^5}{n^4\pi^4}$$

Now multiply both sides of the equation

$$\frac{\ell^5}{5} = \frac{\ell^5}{9} + \sum_{n=1}^{\infty} \frac{8\ell^5}{n^4\pi^4} \quad \text{by } \frac{\pi^4}{8\ell^5} \text{ and obtain } \quad \frac{\pi^4}{40} = \frac{\pi^4}{72} + \sum_{n=1}^{\infty} \frac{1}{n^4}$$

Rearrange this to get

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{40} - \frac{\pi^4}{72} = \frac{\pi^4}{90}$$

Page 145, problem 2. This was problem 5(b) on the first midterm. But let"s try it using the hint. Let

$$\varphi(t) = \|f + tg\|^2 = \langle f + tg, f + tg\rangle = \langle f, f\rangle + 2t\langle f, g\rangle + t^2\langle g, g\rangle.$$

We know that $\varphi(t) \geq 0$. And the minimum of φ happens where $\varphi' = 0$:

$$\varphi'(t) = 2\langle f, g\rangle + 2t\langle g, g\rangle \quad \text{so } \varphi' = 0 \text{ for } \quad t = -\frac{\langle f, g\rangle}{\langle g, g\rangle}.$$

And the minimum is

$$\varphi\left(-\frac{\langle f,g\rangle}{\langle g,g\rangle}\right) = \langle f,f\rangle - 2\frac{\langle f,g\rangle}{\langle g,g\rangle}\langle f,g\rangle + \left(\frac{\langle f,g\rangle}{\langle g,g\rangle}\right)^2\langle g,g\rangle$$
$$= \langle f,f\rangle - 2\frac{\langle f,g\rangle^2}{\langle g,g\rangle} + \frac{\langle f,g\rangle^2}{\langle g,g\rangle}$$
$$= \langle f,f\rangle - \frac{\langle f,g\rangle^2}{\langle g,g\rangle}$$

The minimum value of φ must be non-negative, so

$$\langle f,f\rangle - \frac{\langle f,g\rangle^2}{\langle g,g\rangle} \geq 0 \quad \text{in other words} \quad \langle f,f\rangle\langle g,g\rangle \geq \langle f,g\rangle^2 .$$

Taking the square root of both sides gives the Schwarz inequality.

Page 145, problem 3. A slight variation of this was also on the midterm (this is Poincaré's inequality), but here's a proof of the version given: Use the (square of the) Schwarz inequality with $f = 1$ and $g = f'$, which says that

$$(\langle 1, f'\rangle)^2 \leq \|1\|^2\|f'\|^2$$

Now

$$(\langle 1, f'\rangle)^2 = \left(\int_0^\ell f'(x)\,dx\right)^2 = (f(\ell) - f(0))^2$$

and

$$\|1\|^2\|f'\|^2 = \int_0^\ell 1^2\,dx \int_0^\ell (f'(x))^2\,dx = \ell\int_0^\ell (f'(x))^2\,dx$$

So now Schwarz says

$$(f(\ell) - f(0))^2 \leq \ell\int_0^\ell (f'(x))^2\,dx$$

which is the version of Poincaré that we were to prove.

Page 160, problem 2. If u is a solution of $\Delta u = k^2u$ that depends only on r, then (as we have shown previously, or else from the text) $u(r)$ satisfies

$$u'' + \frac{2}{r}u' = k^2u.$$

Following the hint, let $u = v/r$. Then

$$u' = \frac{v'}{r} - \frac{v}{r^2} \qquad \text{and} \qquad u'' = \frac{v''}{r} - 2\frac{v'}{r^2} + \frac{2v}{r^3}.$$

The equation becomes:

$$\frac{v''}{r} - 2\frac{v'}{r^2} + \frac{2v}{r^3} + \frac{2}{r}\left(\frac{v'}{r} - \frac{v}{r^2}\right) = \frac{v''}{r} = k^2\frac{v}{r}$$

or in other words $v'' = k^2 v$. Since we're given that $k > 0$ (in particular, $k \neq 0$), the solutions of this equation are $v = c_1 e^{kr} + c_2 e^{-kr}$, which in turn yield

$$u(r) = c_1 \frac{e^{kr}}{r} + c_2 \frac{e^{-kr}}{r}.$$

Page 160, problem 5. To solve $\Delta u = 1$ on $r < a$ in $\mathbb{R}^2$ with $u = 0$ when $r = a$, it's enough to look for radial solutions (since we will have shown that the solution of this Dirichlet problem is unique). So we seek a solution $u(r)$ to the ordinary differential equation:

$$u'' + \frac{1}{r} u' = 1$$

that defines a smooth function at the origin and for which $u(a) = 0$. The general solution of

$$u'' + \frac{1}{r} u' = 0 \qquad \text{is} \qquad u_0 = c_1 + c_2 \ln r$$

and a particular solution of the inhomogeneous equation is

$$u_p = \frac{x^2}{4}$$

so the general solution is

$$u_g = \frac{x^2}{4} + c_1 + c_2 \ln r.$$

We don't want a $\ln r$ term since that will become infinite as $r \to 0$, so we set $c_2 = 0$. In order for $u(a) = 0$, we should set $c_1 = -\frac{1}{4} a^2$, and so the solution of the problem is

$$u(r) = \frac{1}{4}(r^2 - a^2) = \frac{1}{4}(x^2 + y^2 - a^2).$$

Page 160, problem 7. The general solution of

$$\Delta u = u_{rr} + \frac{2}{r} u_r = 1$$

is

$$u = c_1 + \frac{c_2}{r} + \frac{r^2}{6}.$$

Now we need to arrange for $u = 0$ when $r = a$ and $r = b$, in other words to solve the linear system

$$\begin{bmatrix} 1 & a^{-1} \\ 1 & b^{-1} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{6} a^2 \\ -\frac{1}{6} b^2 \end{bmatrix}$$

for c_1 and c_2. The solution is

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{b^{-1} - a^{-1}} \begin{bmatrix} b^{-1} & -a^{-1} \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{6} a^2 \\ -\frac{1}{6} b^2 \end{bmatrix} = \frac{ab}{a-b} \begin{bmatrix} \frac{1}{6}(b^2 a^{-1} - a^2 b^{-1}) \\ \frac{1}{6}(a^2 - b^2) \end{bmatrix}$$

Therefore

$$u = \frac{1}{6}\Big(\frac{ab(a+b)}{r} - (b^2 + ab + a^2) + r^2 \Big)$$

Page 164, problem 1. Since this is a pure Neumann problem, it is important to check that the integral of the normal derivative around the boundary of the region is zero, and it is, since the value of the *outward*-pointing normal is $+a$ on a segment of length b (where $x = 0$) and is equal to $-b$ on a segment of length a (where $y = 0$) and zero on the other two segments. So there is a solution to this problem, and it is unique only up to adding a constant.

We could try writing the solution as the sum of two Fourier cosine series:

$$u(x, y) = C + \sum_{n=1}^{\infty} A_n \cosh(a - x) \cos \frac{n\pi y}{b} + \sum_{n=1}^{\infty} B_n \cosh(b - y) \cos \frac{n\pi x}{a}$$

but it is simpler to note that since the boundary conditions are constants, we can seek the solution of the problem as $u(x, y) = F(x) + G(y)$, where $F(x)$ is a quadratic polynomial with $F'(0) = -a$ and $F'(a) = 0$, and where $G(y)$ is also a quadratic polynomial with $G'(0) = b$ and $G(b) = 0$. From the conditions on F, we have that F' is linear in x with slope 1, so $F'(x) = x - a$, and likewise $G'(y) = b - y$. So $F(x) = \frac{1}{2}x^2 - ax$ and $G(x) = by - \frac{1}{2}y^2$ (up to an additive constant), so

$$u(x, y) = F(x) + G(y) + C = \frac{1}{2}x^2 - \frac{1}{2}y^2 - ax + by + C.$$

It's easy to check that this is harmonic and satisfies all the boundary conditions.

Page 164, problem 6. We have to solve the Neumann problem on the cube, with $u_z(x, y, 1) = g(x, y)$ and the normal derivatives zero on the other five sides. So in our separated solutions $X(x)Y(y)Z(z)$ we will have $X'(0) = X'(1) = 0$, $Y'(0) = Y'(1) = 0$ and $Z'(0) = 0$. So our separated solutions are

$$XYZ = \cos m\pi x \cos n\pi y \cosh \sqrt{m^2 + n^2}\pi z$$

and the solution of the problem is

$$u(x, y, z) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} A_{mn} \cos m\pi x \cos n\pi y \cosh \sqrt{m^2 + n^2}\pi z.$$

Now

$$u_z(x, y, 1) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sqrt{m^2 + n^2}\pi \sinh \sqrt{m^2 + n^2}\pi \cos m\pi x \cos n\pi y$$

So, except when $m = n = 0$ (i.e., the constant term)

$$\begin{aligned}
\sqrt{m^2 + n^2}\pi \sinh \sqrt{m^2 + n^2}\pi A_{mn} &= \frac{\langle g(x, y), \cos m\pi x \cos n\pi y\rangle}{\langle \cos m\pi x \cos n\pi y, \cos m\pi x \cos n\pi y\rangle} \\
&= \frac{\int_0^1 \int_0^1 g(x, y) \cos m\pi x \cos n\pi y \, dx \, dy}{\int_0^1 \int_0^1 \cos^2 m\pi x \cos^2 n\pi y \, dx \, dy} \\
&= 4 \int_0^1 \int_0^1 g(x, y) \cos m\pi x \cos n\pi y \, dx \, dy
\end{aligned}$$

and A_{00} is undetermined (and we expect the solution of the Neumann to be unique only up to the addition of an arbitrary constant) . So the solution is

$$u(x,y) = A_{00} + \sum_{\substack{n=0 \\ (n,m)\neq(0,0)}}^{\infty}\sum_{n=0}^{\infty} \frac{4\int_0^1\int_0^1 g(x,y)\cos m\pi x\cos n\pi y\,dx\,dy}{\sqrt{m^2+n^2}\pi\sinh\sqrt{m^2+n^2}\pi}\cos m\pi x\cos n\pi y\cosh\sqrt{m^2+n^2}\pi z$$

Page 172, problem 1. (a) Since $u = 3\sin 2\theta + 1$ on the boundary of the disk, and the maximum of this function of θ is 4, we have that 4 is the maximum value of u throughout the disk by the maximum principle.

(b) The value of u at the origin is the average of u on the boundary of the disk, namely 1.

Page 172, problem 2. We need the harmonic function

$$u = \frac{1}{2}A_0 = \sum_{n=1}^{\infty} r^n(A_n\cos n\theta + B_n\sin n\theta)$$

to equal $1 + 3\sin\theta$ when $r = a$. This will be so if all the $A_n = 0$ except $n = 0$, for which $A_0 = 2$ and all the $B_n = 0$ except for $n = 1$, for which

$$B_1 = \frac{3}{a}.$$

So

$$u = 1 + \frac{3}{a}r\sin\theta.$$

Write up solutions of the following to hand in:

- Page 134 (page 129 in the first ed) problems 14, 15
- Page 145 (page 139 in the first ed) problems 4, 12
- Page 160 (page 154 in the first ed) problems 4, 6, 8
- Page 164 (page 158 in the first ed) problems 4, 7
- *Page 172 (page 163 in the first ed) problem 3

Page 134, problem 14. To find the sum of the series $\sum_{n=1}^{\infty}\frac{1}{n^6}$ we'll find the Fourier series for x^3, using the work from problems 12 and 13 above. From problem 13, we know that the cosine series for x^2 is

$$x^2 \sim \frac{\ell^2}{3} + \sum_{n=1}^{\infty}\frac{(-1)^n 4\ell^2}{n^2\pi^2}\cos\frac{n\pi x}{\ell}$$

so we'll integrate 3 times this to get

$$x^3 = \int_0^x 3t^2\,dt \sim \int_0^x \left[\ell^2 + \sum_{n=1}^{\infty} \frac{(-1)^n 12\ell^2}{n^2\pi^2} \cos\frac{n\pi t}{\ell}\right] dt = \ell^2 x + \sum_{n=1}^{\infty} \frac{(-1)^n 12\ell^3}{n^3\pi^3} \sin\frac{n\pi t}{\ell}\bigg|_0^x$$

$$= \ell^2 x + \sum_{n=1}^{\infty} \frac{(-1)^n 12\ell^3}{n^3\pi^3} \sin\frac{n\pi x}{\ell}$$

And this will do — the series on the right is the series for $x^3 - \ell^2 x$:

$$x^3 - \ell^2 x \sim \sum_{n=1}^{\infty} \frac{(-1)^n 12\ell^3}{n^3\pi^3} \sin\frac{n\pi x}{\ell}$$

and we can apply Parseval's theorem to this:

$$\int_0^\ell (x^3 - \ell^2 x)^2\,dx = \sum_{n=1}^{\infty} \frac{144\ell^6}{n^6\pi^6} \int_0^\ell \sin^2\frac{n\pi x}{\ell}\,dx$$

Since $(x^3 - \ell^2 x)^2 = x^6 - 2\ell^2 x^4 + \ell^4 x^2$, the integral on the left gives $(\frac{1}{7} - \frac{2}{5} + \frac{1}{3})\ell^7 = \frac{8}{105}\ell^7$ And as usual, all the integrals on the right evaluate to $\frac{1}{2}\ell$, so we obtain:

$$\frac{8\ell^7}{105} = \sum_{n=1}^{\infty} \frac{72\ell^7}{n^6\pi^6}$$

Multiply both sides by $\dfrac{\pi^6}{72\ell^7}$ and obtain

$$\frac{\pi^6}{945} = \sum_{n=1}^{\infty} \frac{1}{n^6}.$$

Page 134, problem 15. (a) To write

$$1 = \sum_{n=0}^{\infty} B_n \cos[(n+\tfrac{1}{2})x]$$

we have

$$B_n = \frac{\langle 1, \cos[(n+\frac{1}{2})x]\rangle}{\langle \cos[(n+\frac{1}{2})x], \cos[(n+\frac{1}{2})x]\rangle} = \frac{\displaystyle\int_0^\pi \cos[(n+\tfrac{1}{2})x]\,dx}{\displaystyle\int_0^\pi \cos^2[(n+\tfrac{1}{2})x]\,dx} = \frac{\dfrac{1}{n+\frac{1}{2}} \sin[(n+\frac{1}{2})x]\Big|_0^\pi}{\frac{1}{2}\pi} = \frac{(-1)^n 4}{(2n+1)\pi}$$

(b) The series will converge for all x to the extension of the 1 that is even across $x = 0$, odd across $x\pi$ and 2π-periodic. So, on the interval $-2\pi < x < 2\pi$, the series converges to

$$\begin{cases} -1 & \text{for } -2\pi < x < \pi \\ 0 & \text{for } x = -\pi \\ 1 & \text{for } -\pi < x < \pi \\ 0 & \text{for } x = \pi \\ -1 & \text{for } \pi < x < 2\pi \end{cases}$$

(c) The series is

$$1 = \sum_{n=0}^{\infty} \frac{(-1)^n 4}{(2n+1)\pi} \cos[(n+\tfrac{1}{2})x]$$

for $0 < x < \pi$. And the Parseval's theorem says:

$$\int_0^{\pi} 1^2\,dx = \sum_{n=0}^{\infty} \frac{16}{(2n+1)^2\pi^2} \int_0^{\pi} \cos^2[(n+\tfrac{1}{2})x]\,dx$$

or:

$$\pi = \sum_{n=0}^{\infty} \frac{8}{(2n+1)^2\pi}$$

Multiply both sides by $\frac{1}{8}\pi/$ to get

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}$$

Page 145, problem 4. *Preliminaries*: The boundary conditions for $X(x)$ in the separated solutions are

$$\ell X'(0) + X(0) - X(\ell) = 0 \quad \text{and} \quad \ell X'(\ell) + X(0) - X(\ell) = 0$$

If f and g are two functions that satisfy these boundary conditions, then

$$\begin{aligned} f'(x)g(x) - f(x)g'(x)\Big|_0^{\ell} &= f'(\ell)g(\ell) - f(\ell)g'(\ell) - f'(0)g(0) + f(0)g'(0) \\ &= \frac{1}{\ell}\Big((f(\ell)-f(0))g(\ell) - f(\ell)(g(\ell)-g(0)) - (f(\ell)-f(0))g(0) + f(0)(g(\ell)-g(0))\Big) \\ &= \frac{1}{\ell}\Big(f(\ell)g(\ell) - f(0)g(\ell) - f(\ell)g(\ell) + f(\ell)g(0) - f(\ell)g(0) + f(0)g(0) + f(0)g(\ell) - f(0)g(0)\Big) \\ &= 0 \end{aligned}$$

and so by Theorems 1 and 2 of section 5.3, all the eigenvalues are real and eigenfunctions corresponding to distinct eigenvalues are orthogonal.

We are going to use Poincaré's inequality in part (c) to show that there are no negative eigenvalues, but here is a more direct proof: Suppose $\lambda = -\beta^2$, then

$$X = c_1 \cosh\beta x + c_2 \sinh\beta x \qquad \text{and} \qquad X' = \beta c_1 \sinh\beta x + \beta c_2 \cosh\beta x.$$

The boundary condition $\ell X'(0) + X(0) - X(\ell) = 0$ says

$$\ell\beta c_2 + c_1 - c_1\cosh\beta\ell - c_2\sinh\beta\ell = 0$$

and the boundary condition $\ell X'(\ell) + X(0) - X(\ell) = 0$ says

$$\ell\beta c_1\sinh\beta\ell + \ell\beta c_2\cosh\beta\ell + c_1 - c_2\cosh\beta\ell - c_2\sinh\beta\ell = 0$$

We can simplify things a bit by subtracting the first equation from the second, dividing the result by $\beta\ell$, and replacing the second equation with

$$c_1\sinh\beta\ell + c_2\cosh\beta\ell - c_2 = 0$$

Then the first boundary condition and this last equation comprise the following system of two linear equations in the two unknowns c_1 and c_2:

$$\begin{bmatrix} 1-\cosh\beta\ell & \ell\beta-\sinh\beta\ell \\ \sinh\beta\ell & \cosh\beta\ell-1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

If $-\beta^2$ is to be an eigenvalue, the determinant of the matrix on the left must be zero. The determinant is

$$2\cosh\beta\ell - 1 - \cosh^2\beta\ell + \sinh^2\beta\ell - \beta\ell\sinh\beta\ell = 2\cosh\beta\ell - 2 - \beta\ell\sinh\beta\ell.$$

We need to show there are no nonzero values of β for which this quantity is zero. So we study the function $f(z) = 2\cosh z - 2 - z\sinh z$ We have $f(0) = 0$ and f is an even function. We're going to write the Maclaurin series for f and see that all the coefficients of even powers of z are negative, and all the coefficients of odd powers of z are zero, which will show that $f(z) < 0$ for $z \neq 0$. The series for $\cosh z$ and $\sinh z$ are

$$\cosh z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} \qquad \text{and} \qquad \sinh z = \sum_{n=1}^{\infty} \frac{x^{2n-1}}{(2n-1)!}$$

(just like the series for cosine and sine, but without the alternating signs). So the series for

$$2\cosh z - 2 = \sum_{n=1}^{\infty} \frac{2z^{2n}}{(2n)!}$$

since subtracting 2 cancels the $n = 0$ term, and the series for

$$z\sinh z = \sum_{n=1}^{\infty} \frac{z^{2n}}{(2n-1)!}$$

Therefore

$$2\cosh z - 2 - z\sinh z = \sum_{n=1}^{\infty} \left(\frac{2}{2n} - 1\right) \frac{z^{2n}}{(2n-1)!} = \sum_{n=2}^{\infty} -\frac{n-1}{n} \frac{z^{2n}}{(2n-1)!}$$

which proves the claim that there are no values of z other than $z = 0$ for which this function is zero. Hence there are no negative eigenvalues.

Now we can begin the problem.

(a) Zero is a double eigenvalue of the problem, since any linear function $X(x) = A + Bx$ satisfies the equation $X'' = 0$ together with the boundary conditions, which (as they are printed in the book) say essentially that "the slope equals the slope equals the slope".

Now we can concentrate on the positive eigenvalues and their eigenfunctions. The beginning of the analysis is not so different from the negative case above, except we have trigonometric rather than hyperbolic functions. So, suppose $\lambda = \beta^2$, then

$$X = c_1\cos\beta x + c_2\sin\beta x \qquad \text{and} \qquad X' = -\beta c_1\sin\beta x + \beta c_2\cos\beta x.$$

The boundary condition $\ell X'(0) + X(0) - X(\ell) = 0$ says

$$\ell\beta c_2 + c_1 - c_1\cos\beta\ell - c_2\sin\beta\ell = 0$$

and the boundary condition $\ell X'(\ell) + X(0) - X(\ell) = 0$ says

$$-\ell\beta c_1 \sin\beta\ell + \ell\beta c_2 \cos\beta\ell + c_1 - c_1\cos\beta\ell - c_2\sin\beta\ell = 0$$

We can simplify things a bit by subtracting the first equation from the second, dividing the result by $\beta\ell$, and replacing the second equation with

$$-c_1\sin\beta\ell + c_2\cos\beta\ell - c_2 = 0$$

Then the first boundary condition and this last equation comprise the following system of two linear equations in the two unknowns c_1 and c_2:

$$\begin{bmatrix} 1-\cos\beta\ell & \ell\beta - \sin\beta\ell \\ -\sin\beta\ell & \cos\beta\ell - 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

If β^2 is to be an eigenvalue, the determinant of the matrix on the left must be zero. The determinant is

$$2\cos\beta\ell - 1 - \cos^2\beta\ell - \sin^2\beta\ell + \beta\ell\sin\beta\ell = 2\cos\beta\ell - 2 + \beta\ell\sin\beta\ell.$$

Using the double-angle formulas for sine and cosine, we can rewrite this as

$$-4\sin^2\tfrac{1}{2}\beta\ell + 4(\tfrac{1}{2}\beta\ell)\sin\tfrac{1}{2}\beta\ell\cos\tfrac{1}{2}\beta\ell.$$

So the values of β that give eigenvalues are the roots of

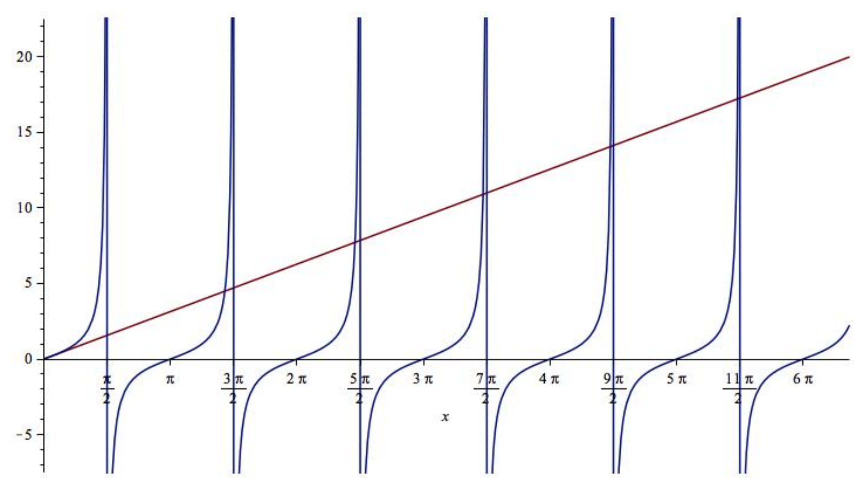
$$\sin\tfrac{1}{2}\beta\ell\left(\tfrac{1}{2}\beta\ell\cos\tfrac{1}{2}\beta\ell - \sin\tfrac{1}{2}\beta\ell\right) = 0$$

If the first factor is zero, i.e., $\sin\frac{1}{2}\beta\ell = 0$, then $\beta = \dfrac{2\pi n}{\ell}$ and the system of equations for c_1 and c_2 becomes:

$$\begin{bmatrix} 0 & 2\pi n \\ 0 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

so that $c_2 = 0$ and the corresponding eigenfunction is $X = c_1\cos\dfrac{2\pi n x}{\ell}$ (for $n = 1, 2, 3, \ldots$).

If the second factor is zero, then $\frac{1}{2}\beta\ell$ must be a root of the equation $x = \tan x$. By graphing $y = x$ and $y = \tan x$ on the same graph and observing where the graphs cross, you can see that the values of $\frac{1}{2}\beta\ell$ that yield eigenvalues are all near (and a bit less than and getting closer and closer to) $(n - \frac{1}{2})\pi$.



In this case we start by rewriting the system of equations for c_1 and c_2 in terms of the angle $\frac{1}{2}\beta\ell$ and the double-angle formulas, since we know that $\sin\frac{1}{2}\beta\ell = \frac{1}{2}\beta\ell\cos\frac{1}{2}\beta\ell$:

$$\begin{bmatrix} 2\sin^2\frac{1}{2}\beta\ell & 2\Big(\frac{1}{2}\beta\ell - \sin\frac{1}{2}\beta\ell\cos\frac{1}{2}\beta\ell\Big) \\ -2\sin\frac{1}{2}\beta\ell\cos\frac{1}{2}\beta\ell & -2\sin^2\frac{1}{2}\beta\ell \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The second equation (after dividing by $-2\sin\frac{1}{2}\beta\ell$) becomes

$$(\cos\tfrac{1}{2}\beta\ell)c_1 + (\sin\tfrac{1}{2}\beta\ell)c_2 = 0$$

Now replace $\sin\frac{1}{2}\beta\ell$ with $\frac{1}{2}\beta\ell\cos\frac{1}{2}\beta\ell$ and divide out the cosine factor to get

$$c_1 + \tfrac{1}{2}\beta\ell c_2 = 0.$$

So the eigenfunction in this case will be (a multiple of)

$$X(x) = \beta\ell\cos\beta x - 2\sin\beta x$$

or, using that $\beta = \sqrt{\lambda}$,

$$X(x) = \sqrt{\lambda}\,\ell\cos\sqrt{\lambda}\,x - 2\sin\sqrt{\lambda}\,x.$$

To summarize, we have three sets of eigenvalues:

- $\lambda = 0$, with eigenfunction $A + Bx$
- $\lambda = \dfrac{4n^2\pi^2}{\ell^2}$ with eigenfunction $\cos\dfrac{2n\pi x}{\ell}$, for $n = 1, 2, 3, \ldots$
- $\lambda_n = \beta_n^2$ where $\frac{1}{2}\beta_n\ell$ are the positive roots of $x = \tan x$ (there are infinitely many of these) with eigenfunctions $X_n(x) = \sqrt{\lambda_n}\,\ell\cos\sqrt{\lambda_n}\,x - 2\sin\sqrt{\lambda_n}\,x$.

So the solution of the heat equation is

$$u(x,t) = A + Bx + \sum_{n=1}^{\infty}\left[P_n e^{-4kn^2\pi^2 t/\ell^2}\cos\frac{2n\pi x}{\ell} + Q_n e^{-k\lambda_n t}\Big(\sqrt{\lambda_n}\,\ell\cos\sqrt{\lambda_n}\,x - 2\sin\sqrt{\lambda_n}\,x\Big)\right]$$

where the P_n and Q_n are the Fourier coefficients of the initial data $\varphi(x)$.

(b) In the solution $u(x,t)$ given above, all the terms of the series have factors that are exponentials of negative numbers times t. So as $t \to \infty$, all these terms go to zero (rapidly!) and we are left with

$$\lim_{t\to\infty} u(x,t) = A + Bx.$$

(c) Using integration by parts (Green's first identity): Suppose $X(x)$ is an eigenfunction with eigenvalue λ; in the integration by parts, let $u = X$ and $dv = X''dx$ so that $du = X'dx$ and $v = X'$:

$$\begin{aligned}
\lambda \langle X, X\rangle &= \langle \lambda X, X\rangle = -\langle X'', X\rangle \\
&= -\int_0^\ell X''(x)X(x)\,dx = -\left[X(x)X'(x)\Big|_0^\ell - \int_0^\ell (X'(x))^2\,dx\right] \\
&= -\left[X(\ell)\left(\frac{X(\ell)-X(0)}{\ell}\right) - X(0)\left(\frac{X(\ell)-x(0)}{\ell}\right)\right] + \int_0^\ell (X'(x))^2\,dx \\
&= \int_0^\ell (X'(x))^2\,dx - \frac{1}{\ell}(X(\ell)-X(0))^2 \\
&\geq 0
\end{aligned}$$

by the result of problem 3. But since $\lambda \langle X, X\rangle \geq 0$ and of course $\langle X, X\rangle > 0$, we must have $\lambda \geq 0$. So there are no negative eigenvalues.

(d) To find A and B, we must be careful to use orthogonal eigenfunctions, and the functions 1 and x are not orthogonal on the interval $0 \leq x \leq \ell$. However, the functions 1 and $x - \frac{1}{2}\ell$ are orthogonal:

$$\left\langle 1, x - \frac{\ell}{2}\right\rangle = \int_0^\ell x - \frac{\ell}{2}\,dx = \frac{x^2}{2} - \frac{\ell x}{2}\Big|_0^\ell = 0$$

Therefore

$$A + Bx = \frac{\langle \varphi, 1\rangle}{\langle 1, 1\rangle} + \frac{\langle \varphi, x - \frac{1}{2}\ell\rangle}{\langle x - \frac{1}{2}\ell, x - \frac{1}{2}\ell\rangle}\left(x - \frac{1}{2}\ell\right)$$

Now $\langle 1, 1\rangle = \ell$, and

$$\left\langle x - \frac{1}{2}\ell, x - \frac{1}{2}\ell\right\rangle = \int_0^\ell \left(x - \frac{\ell}{2}\right)^2 dx = \frac{1}{3}\left(x - \frac{\ell}{2}\right)^3\Bigg|_0^\ell = \frac{2}{3}\left(\frac{\ell}{2}\right)^3 = \frac{\ell^3}{12}$$

Therefore, the components of φ in the directions of 1 and of $x - \frac{1}{2}\ell$ are

$$\frac{\langle \varphi, 1\rangle}{\langle 1, 1\rangle} \qquad \text{and} \qquad \frac{\langle \varphi, x - \frac{1}{2}\ell\rangle}{\langle x - \frac{1}{2}\ell, x - \frac{1}{2}\ell\rangle}$$

and we have $\langle 1, 1\rangle = \int_0^\ell 1^2\,dx = \ell$ and

$$\left\langle x - \frac{1}{2}\ell, x - \frac{1}{2}\ell\right\rangle = \int_0^\ell \left(x - \frac{\ell}{2}\right)^2 dx = \frac{(x - \frac{1}{2}\ell)^3}{3}\Bigg|_0^\ell = \frac{\ell^3}{24} - \frac{(-\ell)^3}{24} = \frac{\ell^3}{12}$$

Therefore

$$\begin{aligned}
Ax+B &= \frac{\langle\varphi,1\rangle}{\langle 1,1\rangle} + \frac{\langle\varphi, x-\frac{1}{2}\ell\rangle}{\langle x-\frac{1}{2}\ell, x-\frac{1}{2}\ell\rangle}\left(x-\frac{\ell}{2}\right) \\
&= \frac{1}{\ell}\int_0^\ell \varphi(x)\,dx + \left(\frac{12}{\ell^3}\int_0^\ell \left(x-\frac{\ell}{2}\right)\varphi(x)\,dx\right)\left(x-\frac{\ell}{2}\right) \\
&= \left[\frac{1}{\ell}\int_0^\ell \varphi(x)\,dx - \frac{\ell}{2}\left(\frac{12}{\ell^3}\int_0^\ell \left(x-\frac{\ell}{2}\right)\varphi(x)\,dx\right)\right] + \left(\frac{12}{\ell^3}\int_0^\ell \left(x-\frac{\ell}{2}\right)\varphi(x)\,dx\right)x \\
&= \int_0^\ell \left(\frac{4}{\ell}-\frac{6x}{\ell^2}\right)\varphi(x)\,dx + \left(\int_0^\ell \left(\frac{12x}{\ell^3}-\frac{6}{\ell^2}\right)\varphi(x)\,dx\right)x
\end{aligned}$$

So

$$A = \int_0^\ell \left(\frac{4}{\ell}-\frac{6x}{\ell^2}\right)\varphi(x)\,dx \quad \text{and} \quad B = \int_0^\ell \left(\frac{12x}{\ell^3}-\frac{6}{\ell^2}\right)\varphi(x)\,dx$$

Page 145, problem 12. Since we know we can integrate a Fourier series term by term but not necessarily differentiate one, start with the series for $f'(x)$:

$$f'(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos nx + B_n \sin nx$$

where

$$A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f'(x)\,dx \quad \text{and} \quad A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f'(x)\cos nx\,dx, \quad B_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f'(x)\sin nx\,dx \quad \text{for } n=1,2,\ldots$$

Because $f(x)$ satisfies periodic boundary conditions, we have

$$0 = f(\pi) - f(-\pi) = \int_{-\pi}^{\pi} f'(x)\,dx = 2\pi A_0$$

So $A_0 = 0$. Next, the Fourier series for $f(x)$ is

$$f(x) = C_0 + \sum_{n=1}^{\infty} C_n \cos nx + D_n \sin nx$$

where

$$C_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx \quad \text{and} \quad C_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx, \quad D_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \quad \text{for } n=1,2,\ldots$$

We are given in the problem that $\int_{-\pi}^{\pi} f(x)\,dx = 0$, so $C_0 = 0$. Also, on integration by parts,

$$C_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{n\pi} f(x)\sin nx\Big|_{-\pi}^{\pi} - \frac{1}{n\pi}\int_{-\pi}^{\pi} f'(x)\sin nx\,dx = -\frac{B_n}{n}$$

and

$$D_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = -\frac{1}{n\pi} f(x)\cos nx\Big|_{-\pi}^{\pi} + \frac{1}{n\pi}\int_{-\pi}^{\pi} f'(x)\cos nx\,dx = \frac{A_n}{n}$$

So Parseval's theorem tells us that (using that the integrals of $\cos nx$ and $\sin nx$ from $-\pi$ to π are equal to π for all n)

$$\int_{-\pi}^{\pi} (f'(x))^2\,dx = \sum_{n=1}^{\infty} (A_n^2 + B_n^2)\pi \geq \sum_{n=1}^{\infty} \left(\frac{A_n^2}{n^2} + \frac{B_n^2}{n^2} \right) \pi = \sum_{n=1}^{\infty} (C_n^2 + D_n^2)\pi = \int_{-\pi}^{\pi} (f(x))^2\,dx.$$

And from this we can also see that equality holds if and only if $A_n = B_n = C_n = D_n = 0$ for all $n \geq 2$.

Page 160, problem 4. We know (from the very first assignment perhaps) that for functions u that depend only on r that

$$\Delta u = u_{rr} + \frac{2}{r} u_r$$

The differential equation $r^2u'' + ru' = 0$ is a Cauchy-Euler equation with general solution $u = c_1 + c_2 r^{-1}$. If $u(a) = A$ and $u(b) = B$, then we have two equations in two unknowns:

$$\begin{bmatrix} 1 & a^{-1} \\ 1 & b^{-1} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} A \\ B \end{bmatrix}$$

the solution of which is

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{b^{-1} - a^{-1}} \begin{bmatrix} b^{-1} & -a^{-1} \\ -1 & 1 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \frac{ab}{a-b} \begin{bmatrix} b^{-1}A + a^{-1}B \\ B - A \end{bmatrix} = \frac{1}{a-b} \begin{bmatrix} aA + bB \\ ab(B-A) \end{bmatrix}$$

so

$$u = \frac{aA + bB}{a-b} + \frac{ab(B-A)}{(a-b)r}$$

Page 160, problem 6. The general solution of

$$\Delta u = u_{rr} + \frac{1}{r} u_r = 1$$

is

$$u = c_1 + c_2 \ln r + \frac{r^2}{4}.$$

Now we need to arrange for $u = 0$ when $r = a$ and $r = b$, in other words to solve the linear system

$$\begin{bmatrix} 1 & \ln a \\ 1 & \ln b \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{4}a^2 \\ -\frac{1}{4}b^2 \end{bmatrix}$$

for c_1 and c_2. The solution is

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{\ln b - \ln a} \begin{bmatrix} \ln b & -\ln a \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{4}a^2 \\ -\frac{1}{4}b^2 \end{bmatrix} = \frac{1}{\ln b - \ln a} \begin{bmatrix} \frac{1}{4}(b^2 \ln a - a^2 \ln b) \\ \frac{1}{4}(a^2 - b^2) \end{bmatrix}$$

Therefore

$$u = \frac{1}{4(\ln b - \ln a)} \Big(b^2 \ln a - a^2 \ln b + (a^2 - b^2) \ln r \Big) + \frac{r^2}{4}$$

Page 160, problem 8. The general solution of

$$\Delta u = u_{rr} + \frac{2}{r} u_r = 1$$

is

$$u = c_1 + \frac{c_2}{r} + \frac{r^2}{6}.$$

Now we need to arrange for $u = 0$ when $r = a$ and for $u_r = 0$ when $r = b$, in other words, to solve the linear system

$$\begin{bmatrix} 1 & a^{-1} \\ 0 & -b^{-2} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{6}a^2 \\ -\frac{1}{3}b \end{bmatrix}$$

for c_1 and c_2. The solution is

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = -b^2 \begin{bmatrix} -b^{-2} & -a^{-1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{6}a^2 \\ -\frac{1}{3}b \end{bmatrix} = -b^2 \begin{bmatrix} \frac{1}{6}(a^2 b^{-2} + 2a^{-1}b) \\ -\frac{1}{3}b \end{bmatrix}$$

Therefore

$$u = \frac{1}{6}\left(\frac{2b^3}{r} - a^2 - 2\frac{b^3}{a} + r^2 \right)$$

As $a \to 0$, the "hole" in the middle of the sphere closes up, and the problem would seem to approach a Neumann problem on the solid ball. Unfortunately, though, the necessary condition for a solution will not be satisfied, since the normal derivative on the surface of the ball is zero, but $\Delta u = 1$ in the interior and the integrals of these can't agree. That is why there is a singularity forming with the a in the denominator of the constant term. The solution will approach $-\infty$ near the $r = b$ boundary.

Page 164, problem 4. To find the harmonic function $u(x, y)$ with the given boundary conditions, we will need to add together harmonic functions v and w that have inhomogeneous conditions on only one side. So let v satisfy

$$v(x, 0) = x \quad v(x, 1) = 0 \quad v_x(0, y) = 0 \quad v_x(1, y) = 0$$

and let w satisfy

$$w(x, 0) = 0 \quad w(x, 1) = 0 \quad w_x(0, y) = 0 \quad w_x(1, y) = y^2.$$

Since v_x is zero for $x = 0$ and $x = 1$, the series for v will have cosines of x, and since v is zero when $y = 1$ but not when $y = 0$, we'll have hyperbolic sines of $1 - y$, (except when $\lambda = 0$, where we have $1 - y$) as follows:

$$v(x, y) = \frac{a_0}{2}(1 - y) + \sum_{n=1}^{\infty} a_n \sinh n\pi(1 - y) \cos n\pi x$$

So for $y = 0$, we'll want

$$v(x, 0) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \sinh n\pi \cos n\pi x = x$$

Therefore, we need

$$a_0 = 2\int_0^1 x\, dx = 1$$

and

$$a_n = \frac{2}{\sinh n\pi}\int_0^1 x\cos n\pi x\,dx = \frac{2}{\sinh n\pi}\left(\frac{x\sin n\pi x}{n\pi} + \frac{\cos n\pi x}{n^2\pi^2}\right)\Bigg|_0^1$$

$$= \begin{cases} 0 & \text{even } n \\ \dfrac{-4}{n^2\pi^2\sinh n\pi} & \text{odd n} \end{cases}$$

Therefore

$$v(x,y) = \frac{1}{2}(1-y) - \sum_{k=0}^{\infty}\frac{4\sinh[(2k+1)\pi(1-y)]}{(2k+1)^2\pi^2\sinh[(2k+1)\pi]}\cos[(2k+1)\pi x]$$

Now w is zero for $y = 0$ and $y = 1$ so the series for w will have sines of y, and since w_x is zero when $x = 0$ but not when $x = 1$, we'll have hyperbolic cosines of x as follows:

$$w(x,y) = \sum_{n=1}^{\infty} b_n \cosh n\pi x \sin n\pi y$$

For $x = 1$, we'll want

$$w_x(1,y) = \sum_{n=1}^{\infty} n\pi b_n \sinh n\pi \sin n\pi y = y^2$$

so we need

$$b_n = \frac{2}{n\pi\sinh n\pi}\int_0^1 y^2\sin n\pi y\,dy = \frac{2}{n\pi\sinh n\pi}\left(-\frac{y^2\cos n\pi y}{n\pi} + \frac{2y\sin n\pi y}{n^2\pi^2} + \frac{2\cos n\pi y}{n^3\pi^3}\right)\Bigg|_0^1$$

$$= \frac{2}{n\pi\sinh n\pi}\left(\frac{(-1)^{n+1}}{n\pi} + \frac{2((-1)^n - 1)}{n^3\pi^3}\right) = \begin{cases} \dfrac{-2}{n^2\pi^2\sinh n\pi} & \text{even } n \\ \dfrac{2}{n^2\pi^2\sinh n\pi} - \dfrac{8}{n^4\pi^4\sinh n\pi} & \text{odd } n \end{cases}$$

Therefore

$$w(x,y) = \sum_{k=1}^{\infty}\left[\frac{2}{(2k-1)^2\pi^2\sinh(2k-1)\pi} - \frac{8}{(2k-1)^4\pi^4\sinh(2k-1)\pi}\right]\cosh[(2k-1)\pi x]\sin[(2k-1)\pi y]$$
$$-\sum_{k=1}^{\infty}\frac{1}{2k^2\pi^2\sinh 2k\pi}\cosh 2k\pi x\sin 2k\pi y$$

So finally,

$$u(x,y) = v(x,y) + w(x,y)$$
$$= \frac{1}{2}(1-y) - \sum_{k=0}^{\infty}\frac{4\sinh[(2k+1)\pi(1-y)]}{(2k+1)^2\pi^2\sinh[(2k+1)\pi]}\cos[(2k+1)\pi x] - \sum_{k=1}^{\infty}\frac{1}{2k^2\pi^2\sinh 2k\pi}\cosh 2k\pi x\sin 2k\pi y$$
$$+ \sum_{k=1}^{\infty}\left[\frac{2}{(2k-1)^2\pi^2\sinh(2k-1)\pi} - \frac{8}{(2k-1)^4\pi^4\sinh(2k-1)\pi}\right]\cosh[(2k-1)\pi x]\sin[(2k-1)\pi y]$$

Page 164, problem 7. Since $u(0,y) = u(\pi,y) = 0$, we'll have $X(0) = X(\pi) = 0$ in the separated solutions, and so $X(x) = \sin nx$ (with eigenvalue n^2). So $Y(y) = ae^{ny} + be^{-ny}$, but because we want the solution to decay to zero as $y \to \infty$, we must have $a = 0$. So the solution is

$$u(x,y) = \sum_{n=1}^{\infty} B_n e^{-ny} \sin nx$$

where

$$B_n = \frac{2}{\pi} \int_0^{\pi} h(x) \sin nx \, dx.$$

(b) If we didn't have the condition at infinity, then we would have

$$u(x,y) = \sum_{n=1}^{\infty} (A_n e^{ny} + B_n e^{-ny}) \sin nx$$

and there would be no way to determine the coefficients, since we would know only that

$$A_n + B_n = \frac{2}{\pi} \int_0^{\pi} h(x) \sin nx \, dx.$$

Page 172, problem 3. Let's start by proving the trig identity for $\sin 3\theta$:

$$\begin{aligned}\sin 3\theta &= \sin(\theta + 2\theta) = \sin\theta\cos 2\theta + \cos\theta \sin 2\theta \\ &= \sin\theta(\cos^2\theta - \sin^2\theta) + 2\cos\theta(\sin\theta\cos\theta) \\ &= 3\sin\theta\cos^2\theta - \sin^3\theta = 3\sin\theta(1 - \sin^2\theta) - \sin^3\theta \\ &= 3\sin\theta - 4\sin^3\theta.\end{aligned}$$

So we have

$$\sin^3\theta = \tfrac{3}{4}\sin\theta - \tfrac{1}{4}\sin 3\theta$$

We need the harmonic function

$$u = \frac{1}{2}A_0 = \sum_{n=1}^{\infty} r^n (A_n \cos n\theta + B_n \sin n\theta)$$

to equal $\frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta$ when $r = a$. This will be so if all the $A_n = 0$ and all the $B_n = 0$ except for $n = 1$ and $n = 3$, for which

$$B_1 = \frac{3}{4a} \quad \text{and} \quad B_3 = -\frac{1}{4a^3}.$$

So

$$u = \frac{3}{4a} r \sin\theta - \frac{1}{4a^3} r^3 \sin 3\theta.$$