Math 425 / AMCS 525 **Dr. DeTurck**
Assignment 8 **Due Tuesday, April 12, 2016**

Topics for this week — Convergence of Fourier series; Laplace's equation and harmonic functions: basic properties, compuations on rectangles and cubes (Fourier!), Poisson's formula for the disk

Eighth Homework Assignment - due Tuesday, April 12

Reading: Read sections 6.4, 7.1, 9.1, 9.2 and 10.1 of the text

Be prepared to discuss the following problems in class:

- Page 175 problems 1, 6
- Page 183 problem 3
- Page 233 problems 2, 3
- Page 240 problems 3,5
- Page 263 problems 1, 5

Page 175, problem 1. For $\Delta u = 0$ outside the circle $r = a$ and u bounded at infinity, we need to use the Fourier series with the r^{-n} terms. Since the boundary data on $r = a$ is $u(a, \theta) = 1 + \sin 3\theta$, which is already a trigonometric polynomial, the series will have just two nonzero terms:

$$u(r, \theta) = 1 + \frac{3a}{r} \sin \theta.$$

Page 175, problem 6. Since $u(r, 0) = u(r, \pi) = 0$, we need only the $r^n \sin n\pi\theta$ terms in the solution of $\Delta u = 0$:

$$u(r, \theta) = \sum_{n=1}^{\infty} B_n r^n \sin n\theta.$$

And to have $u(1, \theta) = \pi \sin \theta - \sin 2\theta$, we should take $B_1 = \pi$, $B_2 = -1$ and the other $B_n = 0$. So the solution is

$$u(r, \theta) = \pi r \sin \theta - r^2 \sin 2\theta.$$

Page 183, problem 3. This is like page 175, problem 11, except the coefficient $a(\mathbf{x})$ is a function rather than a constant and we're in n dimensions now. But the proof proceeds the same way: As usual with uniqueness proofs, let u_1 and u_2 be any two solutions of the problem, and then their difference $v = u_1 - u_2$ satisfies

$$\Delta v = 0 \quad \text{in } D, \qquad \frac{\partial v}{\partial n} + au = 0 \quad \text{on bdy}(D)$$

Write $\Delta v = \nabla \cdot \nabla v$ and recall the identity $\nabla \cdot v\nabla v = v\Delta v + \|\nabla v\|^2$ and calculate:

$$\begin{aligned} 0 = \iiint_D v\Delta v \, d\,\mathrm{vol} &= \iiint_D \nabla \cdot v\nabla v - \|\nabla v\|^2 \, d\,\mathrm{vol} \\ &= \iint_{\mathrm{bd}\, D} v\nabla v \cdot \mathbf{n} \, d\sigma - \iiint_D \|\nabla v\|^2 \, d\,\mathrm{vol} \\ &= \iint_{\mathrm{bd}\, D} v\frac{\partial v}{\partial n} \, d\sigma - \iiint_D \|\nabla v\|^2 \, d\,\mathrm{vol} \\ &= -\iint_{\mathrm{bd}\, D} av^2 \, d\sigma - \iiint_D \|\nabla v\|^2 \, d\,\mathrm{vol} \end{aligned}$$

the last equality because of the boundary condition. Since $a(\mathbf{x}) > 0$ the only way for the last line to be zero is for both integrands to be zero, in which case v is constant (because $\|\nabla v\| = 0$ throughout D), and the constant is zero (because $v = 0$ on the boundary of D). Thus $v(\mathbf{x}) \equiv 0$ so the two solutions u_1 and u_2 of the original problem must be the same, proving uniqueness.

Page 233, problems 2, 3. Let's do these together and in n dimensions. Write $\mathbf{x} = (x_1, x_2, \dots x_n)$. We'll calculate for what exponent p the function

$$u(\mathbf{x}, t) = (c^2t^2 - \mathbf{x}\cdot\mathbf{x})^p$$

satisfies the wave equation (perhaps except on the light cone emanating from the origin, where it will not be defined if $p < 0$). Calculate:

$$\frac{\partial u}{\partial x_i} = -2px_i(c^2t^2 - \mathbf{x}\cdot\mathbf{x})^{p-1} \quad \text{and} \quad \frac{\partial u}{\partial t} = 2pc^2t(c^2t^2 - \mathbf{x}\cdot\mathbf{x})^{p-1}$$

so we get

$$\frac{\partial^2 u}{\partial x_i^2} = -2p(c^2t^2 - \mathbf{x}\cdot\mathbf{x})^{p-1} + 4p(p-1)x_i^2(c^2t^2 - \mathbf{x}\cdot\mathbf{x})^{p-2}$$

and

$$\frac{\partial^2 u}{\partial t^2} = 2pc^2(c^2t^2 - \mathbf{x}\cdot\mathbf{x})^{p-1} + 4p(p-1)c^4t^2(c^2t^2 - \mathbf{x}\cdot\mathbf{x})^{p-2}$$

Therefore

$$\begin{aligned} u_tt - c^2\Delta u &= \left(2pc^2 - c^2\sum_{i=1}^n(-2p)\right)(c^2t^2 - \mathbf{x}\cdot\mathbf{x})^{p-1} + 4p(p-1)\left(c^4t^2 - c^2\sum_{i=1}^n x_i^2\right)(c^2t^2 - \mathbf{x}\cdot\mathbf{x})^{p-2} \\ &= 2pc^2(n+1)(c^2t^2 - \mathbf{x}\cdot\mathbf{x})^{p-1} + 4p(p-1)c^2(c^2t^2 - \mathbf{x}\cdot\mathbf{x})(c^2t^2 - \mathbf{x}\cdot\mathbf{x})^{p-2} \\ &= 2pc^2\Big((n+1) + 2(p-1)\Big)(c^2t^2 - \mathbf{x}\cdot\mathbf{x})^{p-1} \end{aligned}$$

and this will be zero for all $\mathbf{x}$ and t if either $p = 0$ (duh) or $(n+1) + 2(p-1) = 0$, in other words if

$$p = \frac{1-n}{2}$$

and this specializes to the results of problem 2 and 3 for $n = 3$ and 2 respectively. For $n = 1$ this only gives the constant solution, but note that in this case $u(x, t) = \ln(|c^2t^2 - x^2|)$ works.

Page 240, problem 3. We want to solve the three-dimensional wave equation with initial data $u(x, y, z, 0) = 0$ and $u_t(x, y, z, 0) = y$. According to formula (3) in section 9.2, the solution is

$$u(\mathbf{x}, t) = \frac{1}{4\pi c^2 t} \iint_S y \, dS$$

where S is the sphere of radius ct_0 centered at $\mathbf{x}$. Now we don't have to do an actual surface integral here, since it's clear that the *average* value of y on any sphere centered at the point (x_0, y_0, z_0) is y_0, and the area of a sphere of radius ct_0 is $4\pi c^2 t_0^2$. So the integral is equal to $4\pi c^2 t_0^2 y_0$. So

$$u(x, y, z, t) = \frac{4\pi c^2 t^2 y}{4\pi c^2 t} = ty.$$

Page 240, problem 5. By the principle of causality, if the initial data vanish outside the sphere of radius R, then the solution $u(x, t)$ vanishes outside the sphere of radius $R + ct$.

Page 263, problem 1. Homogeneous Neumann conditions implies that the eigenfunctions of the Laplacian on the square take the form $\cos mx \cos ny$ for $m, n \geq 0$. So the general solution of the wave equation $u_{tt} - c^2 \Delta u = 0$ with these boundary conditions is

$$u(x, t) = A_{00} + B_{00} t + \sum_{\substack{m=0 \\ (m,n) \neq (0,0)}}^{\infty} \sum_{n=0}^{\infty} \cos mx \cos ny (A_{mn} \cos \sqrt{m^2 + n^2}\, ct + B_{mn} \sin \sqrt{m^2 + n^2}\, ct)$$

The initial condition $u(x, 0) = 0$ means that all the $A_{mn} = 0$, and $u_t(x, 0) = \sin^2 x = \frac{1}{2}(1 + \cos 2x)$ means that $B_{00} = \frac{1}{2}$, $B_{20} = \frac{1}{4}$ and all the other B_{mn} are zero. So

$$u(x, t) = \frac{1}{2} t + \frac{1}{4} \cos 2x \sin 2t$$

.

Page 263, problem 5. (a) Since $u'' + x^2 u = 0$ is a homogeneous linear second-order ordinary differential equation, its solutions are determined by the choice of initial data (value of u and of u' at one point), so the dimension of the solution space is 2.

(b) The general solution of $u'' + \left(\frac{2\pi}{\ell}\right)^2 u = 0$ is

$$u = c_1 \cos \frac{2\pi t}{\ell} + c_2 \sin \frac{2\pi t}{\ell}$$

and all of these solutions satisfy periodic boundary conditions on the interval $(-\ell, \ell)$, so the dimension of the eigenspace is 2.

(c) The solution of the Neumann problem on the disk (or on any domain) is unique up to adding a constant. So the harmonic functions are the constant functions on the disk, which comprise a vector space of dimension 1.

(d) The Neumann solutions on the square are of the form $\cos m\pi x \cos n\pi y$ for $m, n = 0, 1, 2, \ldots$, and the corresponding eigenvalues are $\lambda_{mn} = (m^2 + n^2)\pi^2$. There are four ways to get $25\pi^2$, namely

$(m,n)=(3,4)$, $(m,n)=(4,3)$, $(m,n)=(5,0)$ and $(m,n)=(0,5)$. So the dimension of this space is 4.

(e) The space of all solutions of the wave equation is the set of functions $u(x,t)$ of the form

$$u(x,t)=f(x+ct)+g(x-ct)$$

for arbitrary functions f and g of one variable, so this space is of infinite dimension.

Write up solutions of the following to hand in:

- Page 175 problems 4, 5, 11
- Page 183 problem 5
- Page 233 problems 1, 5
- Page 240 problems 4, 6
- Page 263 problems 2, 3

Page 175, problem 4. The problem is $\Delta u=0$ outside the circle $r=a$ with $u(a,\theta)=h(\theta)$ and u bounded as $r\to\infty$, so we know that

$$u(r,\theta)=\frac{A_0}{2}+\sum_{n=1}^{\infty}\frac{a^n}{r^n}(A_n\cos n\theta+B_n\sin n\theta)$$

where

$$A_n=\frac{1}{\pi}\int_{-\pi^\pi}h(\theta)\cos n\theta\,d\theta \quad\text{and}\quad B_n=\frac{1}{\pi}\int_{-\pi}^{\pi}h(\theta)\sin n\theta\,d\theta.$$

We jam this all together, changing the variable of integration in the coefficients to φ, to write:

$$\begin{aligned}
u(r,\theta)&=\frac{1}{2\pi}\int_{-\pi}^{\pi}h(\varphi)\,d\varphi+\sum_{n=1}^{\infty}\frac{a^n}{\pi r^n}\left[\left(\int_{-\pi}^{\pi}h(\varphi)\cos n\varphi\,d\varphi\right)\cos n\theta+\left(\int_{-\pi}^{\pi}h(\varphi)\sin n\varphi\,d\varphi\right)\sin n\theta\right]\\
&=\int_{-\pi}^{\pi}h(\varphi)\left[\frac{1}{2\pi}+\sum_{n=1}^{\infty}\frac{a^n}{\pi r^n}(\cos n\varphi\cos n\theta+\sin n\varphi\sin n\theta)\right]d\varphi\\
&=\int_{-\pi}^{\pi}h(\varphi)\left[\frac{1}{2\pi}+\sum_{n=1}^{\infty}\frac{a^n}{\pi r^n}\cos(n\theta-n\varphi)\right]d\varphi\\
&=\int_{-\pi}^{\pi}\frac{h(\varphi)}{2\pi}\left[1+\sum_{n=1}^{\infty}\frac{a^n}{r^n}\left(e^{in(\theta-\varphi)}+e^{-in(\theta-\varphi)}\right)\right]d\varphi
\end{aligned}$$

Now the quantity in the brackets is 1+ the sum of two geometric series; it is

$$1+\sum_{n=1}^{\infty}\left(\frac{ae^{i(\theta-\varphi)}}{r}\right)^n+\sum_{n=1}^{\infty}\left(\frac{ae^{-i(\theta-\varphi)}}{r}\right)^n = 1+\frac{\dfrac{ae^{i(\theta-\varphi)}}{r}}{1-\dfrac{ae^{i(\theta-\varphi)}}{r}}+\frac{\dfrac{ae^{-i(\theta-\varphi)}}{r}}{1-\dfrac{ae^{-i(\theta-\varphi)}}{r}}$$

$$= 1+\frac{ae^{i(\theta-\varphi)}}{r-ae^{i(\theta-\varphi)}}+\frac{ae^{-i(\theta-\varphi)}}{r-ae^{-i(\theta-\varphi)}}$$

$$= 1+\frac{rae^{i(\theta-\varphi)}+rae^{-\theta-\varphi}}{r^2-rae^{i(\theta-\varphi)}-rae^{-e(\theta-\varphi)}+a^2}$$

$$= \frac{r^2-a^2}{r^2-2ra\cos(\theta-\varphi)+a^2}$$

Therefore

$$u(x,t)=\int_{-\pi}^{\pi}\frac{h(\varphi)}{2\pi}\frac{r^2-a^2}{r^2-2ra\cos(\theta-\varphi)+a^2}\,d\varphi$$

which is formula (9).

Page 175, problem 5. We start by writing the general solution for the annulus, which has all the r^n terms as well as the r^{-n} terms:

$$u(r,\theta)=A_0+C_0\ln r+\sum_{n=1}^{\infty}r^n(A_n\cos n\theta+B_n\sin\theta)+r^{-n}(C_n\cos n\theta+D_n\sin n\theta).$$

Now for each part of the problem we have to match the boundary conditions.

(a) The boundary conditions are $u_r(2,\theta)=0$ and $u(1,\theta)=\sin^2\theta=\frac{1}{2}(1-\cos 2\theta)$. So we should have the constant term $A_0=\frac{1}{2}$ (the coefficient of the $\ln r$ term is zero because its derivative at $r=2$ is not zero) and then we have to choose A_2 and C_2 so that for $r=1$ we will have $(A_2+C_2)\cos 2\theta=-\frac{1}{2}\cos 2\theta$ and, since the normal derivative on the outer boundary is

$$u_r(r,\theta)=\sum_{n=1}^{\infty}nr^{n-1}(A_n\cos n\theta+B_n\sin n\theta)-nr^{-n-1}(C_n\cos n\theta+D_n\sin n\theta)$$

for $r=2$ we need $(4A_2-\frac{1}{4}C_2)\cos 2\theta=0$. So A_2 and C_2 satisfy the linear system

$$\begin{bmatrix}1 & 1\\ 4 & -\frac{1}{4}\end{bmatrix}\begin{bmatrix}A_2\\ C_2\end{bmatrix}=\begin{bmatrix}-\frac{1}{2}\\ 0\end{bmatrix}$$

Therefore

$$\begin{bmatrix}A_2\\ C_2\end{bmatrix}=\begin{bmatrix}1 & 1\\ 4 & -\frac{1}{4}\end{bmatrix}^{-1}\begin{bmatrix}-\frac{1}{2}\\ 0\end{bmatrix}=-\frac{4}{17}\begin{bmatrix}-\frac{1}{4} & -1\\ -4 & 1\end{bmatrix}\begin{bmatrix}-\frac{1}{2}\\ 0\end{bmatrix}=-\frac{4}{17}\begin{bmatrix}\frac{1}{8}\\ 2\end{bmatrix}$$

so $A_2=-\frac{1}{34}$ and $C_2=-\frac{8}{17}$. We conclude that

$$u(r,\theta)=\frac{1}{2}-\frac{1}{34}r^2\cos 2\theta-\frac{8}{17}r^{-2}\cos 2\theta.$$

(b) This time, $u(2,\theta) = 0$ and $u(1,\theta) = \frac{1}{2}(1 - \cos 2\theta)$. So we're going to need both A_0 and C_0 as well as A_2 and C_2. Since the constant term at $r = 1$ is just A_0, we have $A_0 = \frac{1}{2}$. But the constant term at $r = 2$ is $A_0 + C_0 \ln 2$, so for this to be zero we need $C_0 = -1/(2\ln 2)$. Next, for the $\cos 2\theta$ terms we need $A_2 + C_2 = -\frac{1}{2}$ at $r = 1$ and $4A_2 + \frac{1}{4}C_2 = 0$ at $r = 2$. So this time, A_2 and C_2 satisfy the linear system

$$\begin{bmatrix} 1 & 1 \\ 4 & \frac{1}{4} \end{bmatrix} \begin{bmatrix} A_2 \\ C_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ 0 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} A_2 \\ C_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 4 & \frac{1}{4} \end{bmatrix}^{-1} \begin{bmatrix} -\frac{1}{2} \\ 0 \end{bmatrix} = -\frac{4}{15} \begin{bmatrix} \frac{1}{4} & -1 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{2} \\ 0 \end{bmatrix} = -\frac{4}{15} \begin{bmatrix} -\frac{1}{8} \\ 2 \end{bmatrix}$$

so $A_2 = \frac{1}{30}$ and $C_2 = -\frac{8}{15}$. We conclude that

$$u(r,\theta) = \frac{1}{2} - \frac{\ln r}{2\ln 2} + \frac{r^2}{30}\cos 2\theta - \frac{8}{15r^2}\cos 2\theta.$$

Page 175, problem 11. As usual with uniqueness proofs, let u_1 and u_2 be any two solutions of the problem, and then their difference $v = u_1 - u_2$ satisfies

$$\Delta v = 0 \quad \text{in } D, \qquad \frac{\partial v}{\partial n} + au = 0 \quad \text{on bdy}(D)$$

Write $\Delta v = \nabla \cdot \nabla v$ and recall the identity $\nabla \cdot v\nabla v = v\Delta v + \|\nabla v\|^2$ and calculate:

$$\begin{aligned} 0 = \iiint_D v\Delta v \, d\,\text{vol} &= \iiint_D \nabla \cdot v\nabla v - \|\nabla v\|^2 \, d\,\text{vol} \\ &= \iint_{\text{bd } D} v\nabla v \cdot \mathbf{n} \, d\sigma - \iiint_D \|\nabla v\|^2 \, d\,\text{vol} \\ &= \iint_{\text{bd } D} v\frac{\partial v}{\partial n} \, d\sigma - \iiint_D \|\nabla v\|^2 \, d\,\text{vol} \\ &= -\iint_{\text{bd } D} av^2 \, d\sigma - \iiint_D \|\nabla v\|^2 \, d\,\text{vol} \end{aligned}$$

the last equality because of the boundary condition. Since $a > 0$ the only way for the last line to be zero is for both integrands to be zero, in which case v is constant (because $\|\nabla v\| = 0$ throughout D), and the constant is zero (because $v = 0$ on the boundary of D). Thus $v(x,y,z) \equiv 0$ so the two solutions u_1 and u_2 of the original problem must be the same, proving uniqueness.

Page 183, problem 5 Let u be a harmonic function in D (which is unique up to adding a constant) that satisfies

$$\frac{du}{dn} = h(\mathbf{x}) \quad \text{on } \text{bd}(D)$$

and let v be any function on D. Using the divergence theorem and the identity $\nabla \cdot g\nabla f = g\Delta f +$

$\nabla g \cdot \nabla f$ we can conclude:

$$\begin{aligned}
E[u+v] &= \frac{1}{2}\iiint_D (\nabla u + \nabla v)\cdot(\nabla u + \nabla v)\,d\mathbf{x} - \iint_{\mathrm{bd}(D)} hu + hv\,dS \\
&= E[u] + \frac{1}{2}\iiint_D 2\nabla u\cdot\nabla v + \nabla v\cdot\nabla v\,d\mathbf{x} - \iint_{\mathrm{bd}(D)} hv\,dS \\
&= E[u] + \left(\iiint_D \nabla u\cdot\nabla v\,d\mathbf{x} - \iint_{\mathrm{bd}(D)} v\nabla u\cdot\mathbf{n}\,dS\right) + \frac{1}{2}\iiint_D \nabla v\cdot\nabla v\,d\mathbf{x} \\
&= E[u] - \iiint_D v\Delta u\,d\mathbf{x} + \frac{1}{2}\iiint_D \nabla v\cdot\nabla v\,d\mathbf{x} \\
&= E[u] + \frac{1}{2}\iiint_D \nabla v\cdot\nabla v\,d\mathbf{x} \\
&\geq E[u]
\end{aligned}$$

(to get the next-to-last lines we used that u is a solution of the Neumann problem). This shows that the energy functional is minimized when u is a solution of the Neumann problem, and that $E[u+v] = E[u]$ exactly when v is a constant.

Page 233, problem 1. We might as well do this one in n space dimensions, so suppose that $\mathbf{x} = (x_1, x_2, \ldots x_n)$ and $\mathbf{k} = (k_1, k_2, \ldots, k_n)$. If $u(\mathbf{x}, t) = f(\mathbf{k}\cdot\mathbf{x} - ct)$, then

$$\frac{\partial u}{\partial x_i} = k_i f'(\mathbf{k}\cdot\mathbf{x} - ct) \quad \text{and} \quad \frac{\partial u}{\partial t} = -cf'(\mathbf{k}\cdot\mathbf{x} - ct)$$

and

$$\frac{\partial^2 u}{\partial x_i^2} = k_i^2 f''(\mathbf{k}\cdot\mathbf{x} - ct) \quad \text{and} \quad \frac{\partial^2 u}{\partial t^2} = c^2 f'(\mathbf{k}\cdot\mathbf{x} - c^t).$$

If $u_{tt} - c^2\Delta u = 0$, then

$$(c^2 - c^2\mathbf{k}\cdot\mathbf{k})f''(\mathbf{k}\cdot\mathbf{x} - ct) = 0$$

so either f is linear and $\mathbf{k}$ is arbitrary, or else $\mathbf{k}\cdot\mathbf{k} = 1$ and f is arbitrary. (This squares with what we already know about the $n = 1$ case)

Page 233, problem 5. Causality means that for some time $t_1 > 0$, the values of the solution u at points in the disk centered at $\mathbf{x}_0 = (x_0, y_0)$ with radius r_1 depend only on the initial data in the disk centered at $\mathbf{x}_0$ with radius $r_0 = r_1 + ct_1$. Equivalently, if the initial data (position and velocity) are identically zero in the disk of radius $r_1 + ct_1$ centered at $\mathbf{x}_0$, then the solution is identically zero at time t_1 in the disk of radius r_1 centered at $\mathbf{x}_0$. To do this, we'll show that the energy

$$E[t_1] = \iint_{D_1} \tfrac{1}{2}u_t^2 + \tfrac{1}{2}c^2|\nabla u|^2\,dx\,dy$$

where D_1 is the disk of radius r_1 in the $t = t_1$ plane centered at $(\mathbf{x}_0, t_1) = (x_0, y_0, t_1)$ is no larger than the energy

$$E[0] = \iint_{D_0} \tfrac{1}{2}u_t^2 + \tfrac{1}{2}c^2|\nabla u|^2\,dx\,dy$$

where D_0 is the disk of radius $r_0 r_1 + ct_1$ in the $t = 0$ plane centered at $(\mathbf{x}_0, 0) = (x_0, y_0, 0)$. So if the energy $E[0] = 0$, then the energy $E[t_1]$ must also be zero, since the energy is clearly a non-negative function of t, which in turn will imply that the solution is identically zero in the disk D_1 at time t_1.

To prove this energy inequality, we are going to apply the divergence theorem to a region determined by the values x_0, y_0, t_1 and r_1. We'll let $t_2 = t_1 + \dfrac{r_1}{c}$ so that the cone in xyt-space with vertex $(\mathbf{x}_0, t_2)$ and base in the $t = 0$ plane being the disk centered at $(x_0, 0)$ with radius $r_0 = ct_2 = r_1 + ct_1$ has the initial disk and the time t_1 disk as cross-sections.

We'll need to use the vector calculus identity:

$$\nabla \cdot (v\nabla u) = v\Delta u + \nabla v \cdot \nabla u$$

with $v = u_t$ to get the "energy identity"

$$(\tfrac{1}{2}u_t^2 + \tfrac{1}{2}c^2\|\nabla u\|^2)_t = u_t u_{tt} + c^2\nabla u \cdot \nabla u_t = u_t u_{tt} - c^2 u_t \Delta u + c^2\nabla \cdot (u_t \nabla u) = c^2\nabla \cdot (u_t\nabla u)$$

if u satisfies the wave equation $u_{tt} = c^2\Delta u$. Note that the gradients, divergences and Laplacien are taken in 2-dimensional xy-space. A consequence of this is that the *three*-dimensional divergence of the three-dimensional vector (in xyt-space):

$$\mathbf{V} = [-c^2 u_t u_x\,,\, -c^2 u_t u_y\,,\, \tfrac{1}{2}u_t^2 + \tfrac{1}{2}c^2\|\nabla u\|^2] = [-c^2 u_t \nabla u\,,\, \tfrac{1}{2}u_t^2 + \tfrac{1}{2}c^2\|\nabla u\|^2]$$

is zero. So the surface integral of $\mathbf{V} \cdot \mathbf{n}\, d\sigma$ over the boundary of any solid region in xyt-space is zero.

The solid region to which we're going to apply the preceding to is constructed as in the first paragraph above: Starting from the point $(\mathbf{x}_0, t_2)$ (assume $t_2 > 0$, we consider the cone with vertex $(\mathbf{x}_0, t_2)$ and with base in the $t = 0$ plane being the disk centered at $(\mathbf{x}_0, 0)$ and with radius ct_2. So the lateral surface of the cone is given by the equation

$$(x - x_0^2) + (y - y_0)^2 = c^2(t_0 - t)^2$$

which we can rewrite (letting $\mathbf{x} = (x, y)$) as

$$\|\mathbf{x} - \mathbf{x}_0\|^2 - c^2(t_0 - t)^2 = 0.$$

The solid we're going to consider is a "frustum" F of this cone between the two parallel planes $t = 0$ and $t = t_1$ where $0 < t_1 < t_2$ The surface of F has three parts: the top, which is the disk of radius $c(t_2 - t_1)$ centered at $(\mathbf{x}_0, t_1)$ in the plane $t = t_1$, the bottom, which is the base of the cone, and the side, which is the graph of the equation of the cone for $0 < t < t_1$.

We observe that since the outward-pointing normal from $\mathrm{bd}(F)$ on the top $(t = t_1)$ disk is the unit vector $[0, 0, 1]$ pointing in the positive t direction, the part of the surface integral of $\mathbf{V} \cdot \mathbf{n}\, d\sigma$ over the top surface is precisely the energy $E[t_1]$. Likewise, the outward pointing normal from $\mathrm{bd}(F)$ on the bottom $(t = 0)$ disk is the unit vector $[0, 0, -1]$ pointing in the negative t diversion, so that the part of the surface integral of $\mathbf{V} \cdot \mathbf{n}\, d\sigma$ over the bottom surface is the negative of the energy at time $t = 0$, i.e., it is $-E[0]$.

For the integral over the side, we need the outward pointing normal, which is the (normalized) gradient of the defining equation:

$$\|\mathbf{x} - \mathbf{x}_0\|^2 - c^2(t_0 - t)^2 = 0,$$

in other words, it is the normalized version of

$$\mathbf{N} = [2(\mathbf{x} - \mathbf{x}_0)\,,\, 2c^2(t_0 - t)]$$

Now we calculate:

$$\mathbf{N} \cdot \mathbf{V} = c^2\Big((t_0 - t)(u_t^2 + c^2\|\nabla u\|^2) - 2u_t(\mathbf{x} - \mathbf{x}_0) \cdot \nabla u\Big)$$

Let $r = \|\mathbf{x} - \mathbf{x}_0\| = c(t_0 - t)$. Then we can factor $(t_0 - t)$ out and get:

$$\begin{aligned}
\mathbf{N} \cdot \mathbf{V} &= cr\Big(u_t^2 + c^2\|\nabla u\|^2 - 2cu_t \frac{\mathbf{x} - \mathbf{x}_0}{r} \cdot \nabla u\Big) \\
&= cr\left(u_t^2 - 2cu_t \frac{\mathbf{x} - \mathbf{x}_0}{r} \cdot \nabla u + \left(c\frac{\mathbf{x} - \mathbf{x}_0}{r} \cdot \nabla u\right)^2 - \left(c\frac{\mathbf{x} - \mathbf{x}_0}{r} \cdot \nabla u\right)^2 + c^2\|\nabla u\|^2\right) \\
&= cr\left(\left(u_t - c\frac{\mathbf{x} - \mathbf{x}_0}{r} \cdot \nabla u\right)^2 + c^2\|\nabla u\|^2 - \left(c\frac{\mathbf{x} - \mathbf{x}_0}{r} \cdot \nabla u\right)^2\right)
\end{aligned}$$

The sum of the last two terms in the parentheses is non-negative, since $r = \|\mathbf{x} - \mathbf{x}_0\|$, so $(\mathbf{x} - \mathbf{x}_0)/r$ is a unit vector, which means that the last term is $c^2\|\nabla u\|^2 \cos^2\theta$ where θ is the angle between ∇u and $\mathbf{x} - \mathbf{x}_0$. So the whole expression adds up to something non-negative, which means that the integral of $\mathbf{N} \cdot \mathbf{V}\, d\sigma$ over the side of the cone is non-negative.

Summing up the three parts, we have

$$E[t_1] - E[0] + \text{something non-negative} = 0$$

which implies that $E[0] \geq E[t_1]$, which is just what we needed to show. Therefore the energy (and in turn u) will be zero in the disk D_1 if it is zero in the disk D_0.

Page 240, problem 4. The Laplacian in spherical coordinates is

$$\Delta u = u_{\rho\rho} + \frac{2}{\rho}u_\rho + \frac{1}{\rho^2}\left(u_{\theta\theta} + (\cot\theta)u_\theta + \frac{1}{\sin^2\theta}u_{\varphi\varphi}\right)$$

But since the initial data for this problem depend only on ρ, namely $u(\mathbf{x}, 0) = 0$ and $u_t(\mathbf{x}, 0) = \rho^2$, we expect the solution to depend only on ρ (and t). Thus we seek the solution $u(\rho, t)$ of

$$u_{tt} = c^2\left(u_{\rho\rho} + \frac{2}{\rho}u_\rho\right) \qquad u(\rho, t) = 0 \quad u_t(\rho, t) = \rho^2.$$

But then, following the derivation of Kirchhoff's formula we know that the function $v(\rho, t) = \rho u(\rho, t)$ satisfies

$$v_{tt} = c^2 v_{\rho\rho} \qquad v(\mathbf{x}, 0) = 0 \quad v_t(\mathbf{x}, 0) = \rho^3.$$

But now, by d'Alembert's formula,

$$\begin{aligned}
v(\rho, t) &= \frac{1}{2c}\int_{\rho - ct}^{\rho + ct} s^3\, ds = \frac{1}{8c}\left((\rho + ct)^4 - (\rho - ct)^4\right) \\
&= \frac{1}{c}\left(\rho^3 ct + \rho c^3 t^3\right) = \rho^3 t + \rho c^2 t^3
\end{aligned}$$

Therefore

$$u(\rho, t) = \frac{1}{\rho}v(\rho, t) = \rho^2 t + c^2 t^3 = x^2 t + y^2 t + z^2 t + c^2 t^3$$

Page 240, problem 6. (a) S is the sphere with center at $\mathbf{x}$ and radius R — for convenience, let's put the point $\mathbf{x}$ on the z-axis at $z = a$ (and assume that $a > 0$). We want the surface area of the part of S that lies inside the sphere Q of radius ρ centered at the origin. First, we'll assume that $a > \rho$, so the point $\mathbf{x}$ is outside the sphere Q. It's easy to see that if $R < a - \rho$, then the spheres S and

Q will not intersect. Nor will they intersect if $R > a + \rho$. So the surface area is zero in both these situations. If $a - \rho < R < a + \rho$, then the intersection of the two spheres will be a circle parallel to the xy-plane. What we need to find is the z-coordinate of the circle of intersection. We can restrict our attention to the points in the yz-plane, and find the intersection of the two circles S_{yz} centered at $(0, 0, a)$ with radius R and Q_{yz} centered at $(0, 0, 0)$ with radius ρ. So for some value of the angles θ and φ, the coordinates of the point (well, points, since there are two of them) of intersection are

$$(0, \rho\cos\theta, \rho\sin\theta) \quad \text{(as seen from } Q_{yz}\text{)}$$

and

$$(0, a - R\cos\varphi, R\sin\varphi) \quad \text{(as seen from } S_{yz}\text{)}.$$

These are the coordinates of the same point, so we have the equations

$$\rho\cos\theta = a - R\cos\varphi \qquad \text{and} \qquad \rho\sin\theta = R\sin\varphi.$$

Square the equations and add them together to get

$$\rho^2 = a^2 + R^2 - 2Ra\cos\varphi$$

or

$$R\cos\varphi = \frac{a^2 + R^2 - \rho^2}{2a}$$

Now the z-coordinate of the point of intersection was $a - R\cos\varphi$, so we conclude that we're interested in the area of the part of S between the parallel planes $z = a - R$ and $z = a - \dfrac{a^2 + R^2 - \rho^2}{2a}$. Now, a remarkable fact about spheres (that you should prove for yourself if you've never done it) is that the surface area of the part of a sphere between two parallel planes that intersect (or are tangent to) it is equal to the lateral surface area of a cylinder with the same radius as the sphere and height equal to the distance between the parallel planes. So the surface area we're interested in is

$$2\pi R\left[\left(a - \frac{a^2 + R^2 - \rho^2}{2a}\right) - (a - R)\right] = 2\pi R\frac{2aR - a^2 - R^2 + \rho^2}{2a} = \frac{\pi R}{a}\left(\rho^2 - (a - R)^2\right)$$

Now this formula is valid for any $\mathbf{x}$ outside the sphere Q with $|\mathbf{x}|$ replacing the parameter a. So for $\mathbf{x}$ outside of Q we have

$$SA = \begin{cases} 0 & \text{for } R < |\mathbf{x}| - \rho \\ \dfrac{\pi R}{|\mathbf{x}|}\left(\rho^2 - (|\mathbf{x}| - R)^2\right) & \text{for } |\mathbf{x}| - \rho < R < |\mathbf{x}| + \rho \\ 0 & \text{for } R > |\mathbf{x}| + \rho \end{cases}$$

Next, if $a < \rho$, so that the point $\mathbf{x}$ is inside the sphere Q of radius ρ centered at the origin. Then for $R \le \rho - a$ the entire sphere of radius R centered at $\mathbf{x}$ is contained in the sphere Q, so the surface area is $4\pi R^2$. For $\rho - a \le R \le \rho + a$, we have the surface area of the part of the sphere of radius R centered at $\mathbf{x}$ between the planes $z = a - R$ and $z = a - \dfrac{a^2 + R^2 - \rho^2}{2a}$ as before. so in this case:

$$SA = \begin{cases} 4\pi R^2 & \text{for } R < \rho - |\mathbf{x}| \\ \dfrac{\pi R}{|\mathbf{x}|}\left(\rho^2 - (|\mathbf{x}| - R)^2\right) & \text{for } \rho - |\mathbf{x}| < R < |\mathbf{x}| + \rho \\ 0 & \text{for } R > |\mathbf{x}| + \rho \end{cases}$$

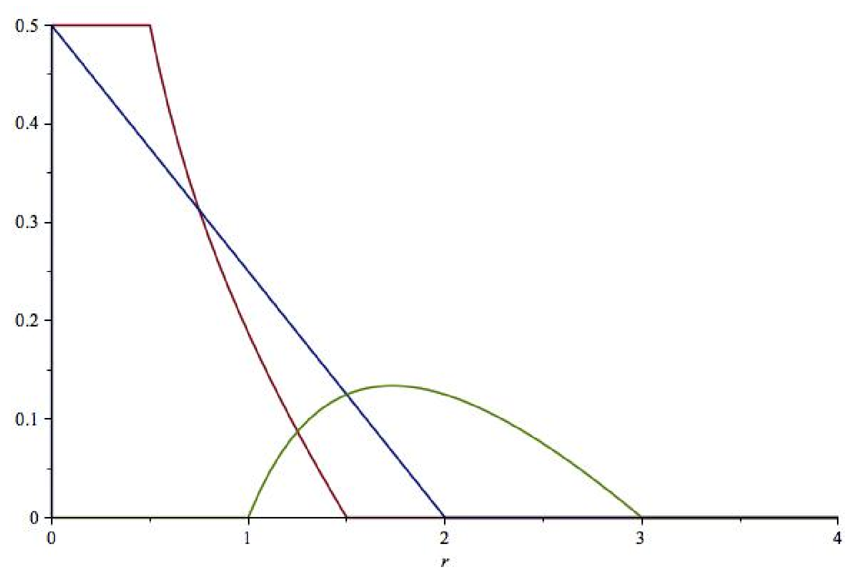
(b) According to Kirchhoff's formula, to find $u(\mathbf{x}, t)$, we need to set $R = ct$ in the surface area formulas above, multiply them by A, and divide by $4\pi c^2 t$. So we get:

$$\text{If } |\mathbf{x}| > \rho: \qquad u(\mathbf{x}, t) = \begin{cases} 0 & \text{for } ct < |\mathbf{x}| - \rho \\ \dfrac{A}{4c|\mathbf{x}|}\Big(\rho^2 - (|\mathbf{x}| - ct)^2\Big) & \text{for } |\mathbf{x}| - \rho < ct < |\mathbf{x}| + \rho \\ 0 & \text{for } ct > |\mathbf{x}| + \rho \end{cases}$$

and

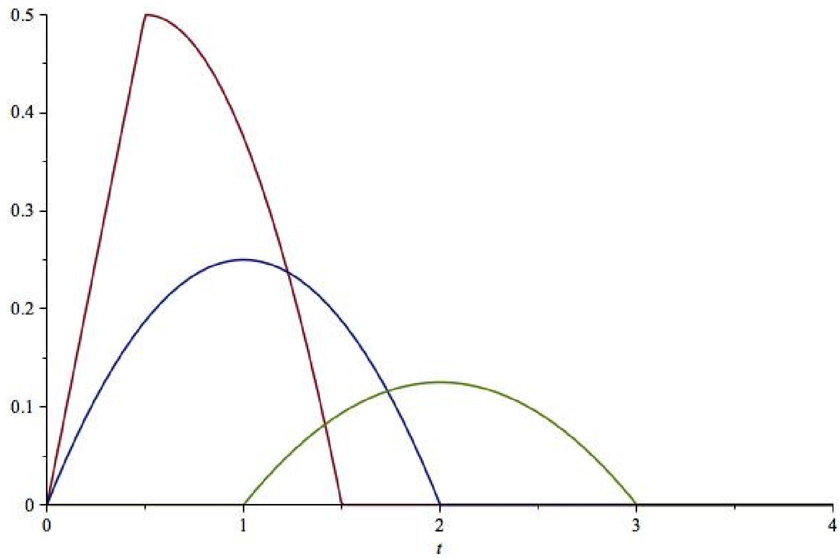
$$\text{If } |\mathbf{x}| < \rho: \qquad u(\mathbf{x}, t) = \begin{cases} At & \text{for } ct < |\mathbf{x}| - \rho \\ \dfrac{A}{4c|\mathbf{x}|}\Big(\rho^2 - (|\mathbf{x}| - ct)^2\Big) & \text{for } |\mathbf{x}| - \rho < ct < |\mathbf{x}| + \rho \\ 0 & \text{for } ct > |\mathbf{x}| + \rho \end{cases}$$

(c)



In the figure, the profile for $t = 0.5$ is red, for $t = 1$ is blue and for $t = 2$ is green.

(d)



In the figure, the path for $r = 0.5$ is red, for $r = 1$ is blue and for $r = 2$ is green.

(e) First of all, if $|\mathbf{x}_0| < \rho$ and $|\mathbf{v}| = c$, then

$$|\mathbf{x}_0 + t\mathbf{v}| \leq |\mathbf{x}_0| + t|\mathbf{v}| < \rho + ct$$

and

$$|\mathbf{x}_0 + t\mathbf{v}| \geq t|\mathbf{v}| - |\mathbf{x}_0| > ct - \rho$$

so for all t the value of $u(\mathbf{x}_0 + t\mathbf{v}, t)$ is determined by the middle (complicated) line in the formula for u given in part (b). So the problem is to calculate:

$$\begin{aligned}
\lim_{t \text{ø} \infty} t \cdot u(\mathbf{x}_0 + t\mathbf{v}, t) &= \lim_{t\to\infty} \frac{At}{4c|\mathbf{x}_0 + t\mathbf{v}|} \left(\rho^2 - (|\mathbf{x}_0 + t\mathbf{v}| - ct)^2\right) \\
&= \lim_{t\to\infty} \frac{A}{4c\left|\dfrac{\mathbf{x}_0}{t} + \mathbf{v}\right|} \left(\rho^2 - \left(\frac{(|\mathbf{x}_0 + t\mathbf{v}| - ct)(|\mathbf{x}_0 + t\mathbf{v}| + ct)}{(|\mathbf{x}_0 + t\mathbf{v}| + ct)}\right)^2\right) \\
&= \frac{A}{4c|\mathbf{v}|} \lim_{t\to\infty} \left(\rho^2 - \left(\frac{|\mathbf{x}_0 + t\mathbf{v}|^2 - c^2t^2}{|\mathbf{x}_0 + t\mathbf{v}| + ct}\right)^2\right) \\
&= \frac{A}{4c^2} \lim_{t\to\infty} \left(\rho^2 - \left(\frac{(\mathbf{x}_0 + t\mathbf{v}) \cdot (\mathbf{x}_0 + t\mathbf{v}) - t^2|\mathbf{v}|^2}{|\mathbf{x}_0 + t\mathbf{v}| + ct}\right)^2\right) \\
&= \frac{A}{4c^2} \lim_{t\to\infty} \left(\rho^2 - \left(\frac{|\mathbf{x}_0|^2 + 2t\mathbf{x}_0 \cdot \mathbf{v}}{|\mathbf{x}_0 + t\mathbf{v}| + t|\mathbf{v}|}\right)^2\right) \\
&= \frac{A}{4c^2} \lim_{t\to\infty} \left(\rho^2 - \left(\frac{\dfrac{|\mathbf{x}_0|^2}{t} + 2\mathbf{x}_0 \cdot \mathbf{v}}{\left|\dfrac{\mathbf{x}_0}{t} + \mathbf{v}\right| + |\mathbf{v}|}\right)^2\right) \\
&= \frac{A}{4c^2} \left(\rho^2 - \frac{(\mathbf{x}_0 \cdot \mathbf{v})^2}{|\mathbf{v}|^2}\right)
\end{aligned}$$

Page 263, problem 2. The Dirichlet boundary conditions imply that the eigenfunctions of the Laplacian on the square take the form $\sin mx \sin ny$ for $m, n > 0$. So the general solution of the wave equation $u_{tt} - c^2 \Delta u = 0$ with these boundary conditions is

$$u(x,t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \left(A_{mn} \cos\sqrt{\frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2}}\, ct + B_{mn} \sin\sqrt{\frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2}}\, ct \right)$$

The initial conditions $u_t(x, y, 0) = 0$ implies that all the B_{mn} are zero. The condition

$$u(x, y, 0) = xy(a - x)(b - y)$$

means that

$$A_{mn} = \frac{\left\langle xy(a-x)(b-y), \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \right\rangle}{\left\langle \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}, \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \right\rangle}$$

The denominator is

$$\left\langle \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}, \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \right\rangle = \int_0^b \int_0^a \sin^2\frac{m\pi x}{a} \sin^2\frac{n\pi y}{b}\, dy\, dx = \frac{ab}{4}$$

and the numerator is

$$\begin{aligned} \left\langle xy(a-x)(b-y), \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \right\rangle &= \int_0^b \int_0^a xy(a-x)(b-y) \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}\, dy\, dx \\ &= \int_0^a x(a-x) \sin\frac{m\pi x}{a}\, dx \int_0^b y(b-y) \sin\frac{n\pi y}{b}\, dy \end{aligned}$$

Now

$$\begin{aligned} \int_0^a x(a-x) \sin\frac{m\pi x}{a}\, dx &= -\frac{a}{m\pi} x(a-x) \cos\frac{m\pi x}{a}\bigg|_0^a + \frac{a}{m\pi} \int_0^a (a-2x) \cos\frac{m\pi x}{a}\, dx \\ &= \frac{a^2}{m^2\pi^2} (a-2x) \sin\frac{m\pi x}{a}\bigg|_0^a + \frac{2a^2}{m^2\pi^2} \int_0^a \sin\frac{m\pi x}{a}\, dx \\ &= -\frac{2a^3}{m^3\pi^3} \cos\frac{m\pi x}{a}\bigg|_0^a = \begin{cases} \dfrac{4a^3}{m^3\pi^3} & \text{if } m \text{ is odd} \\ 0 & \text{if } m \text{ is even} \end{cases} \end{aligned}$$

So likewise

$$\int_0^b y(b-y) \sin\frac{n\pi y}{b}\, dy = \begin{cases} \dfrac{4b^3}{n^3\pi^3} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

Letting $m = 2k - 1$ and $n = 2\ell - 1$ we can write

$$A_{k\ell} = \frac{16a^3b^3}{(2k-1)^3(2\ell-1)^3\pi^6}$$

for the coefficient of $\sin\frac{2k\pi x}{a} \sin\frac{2\ell\pi y}{b}$ and conclude

$$u(x,t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{16a^3b^3}{(2k-1)^3(2\ell-1)^3\pi^6} \sin\frac{(2k-1)\pi x}{a} \sin\frac{(2\ell-1)\pi y}{b} \cos\sqrt{\frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2}}\, ct$$

Page 263, problem 3. Since $u_t = k\Delta u + \gamma u$, with Dirichlet boundary conditions, we know that each term in the series solution of the problem will have a factor of the form $e^{-\lambda kt}$, where λ is an eigenvalue of $\Delta + \dfrac{\gamma}{k}$ on the cube, in other words,

$$\Delta u + \left(\frac{\gamma}{k} + \lambda\right) u = 0.$$

The eigenfunctions are

$$\varphi_{\ell mn}(x,y,z) = \sin\frac{\ell\pi x}{a}\sin\frac{m\pi y}{a}\sin\frac{n\pi z}{a} \quad \text{for } \ell, m.n = 1,2,3,\ldots$$

as usual, but the eigenvalue corresponding to $\varphi_{\ell mn}$ is

$$\lambda_{\ell mn} = \frac{\ell^2\pi^2}{a^2} + \frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{a^2} - \frac{\gamma}{k}.$$

We need the smallest eigenvalue, namely λ_{111} to be non-negative, so this means that

$$\gamma \leq \frac{3k\pi^2}{a^2}.$$