Math 425 / AMCS 525 **Dr. DeTurck**

Assignment 9 **Due Tuesday, April 19, 2016**

1. Prove formulas (1)–(6) concerning Bessel functions.

For formula (1), we have

$$\begin{aligned}
\frac{d}{dx}(x^n J_n(x)) &= \frac{d}{dx}\left(\sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+n)!} \frac{x^{2n+2k}}{2^{n+2k}}\right) \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+n)!} \frac{(2n+2k)x^{2n+2k-1}}{2^{n+2k}} \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k(n+k)}{k!(k+n)!} \frac{x^{2n+2k-1}}{2^{n-1+2k}} \\
&= x^n \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+n-1)!} \frac{x^{n+2k-1}}{2^{n-1+2k}} \\
&= x^n J_{n-1}(x).
\end{aligned}$$

Similarly, for formula (2),

$$\begin{aligned}
\frac{d}{dx}(x^{-n} J_n(x)) &= \frac{d}{dx}\left(\sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+n)!} \frac{x^{2k}}{2^{n+2k}}\right) \\
&= \sum_{k=1}^{\infty} \frac{(-1)^k}{k!(k+n)!} \frac{(2k)x^{2k-1}}{2^{n+2k}} \\
&= \sum_{k=1}^{\infty} \frac{(-1)^k k}{k!(k+n)!} \frac{x^{-n}x^{n+2k-1}}{2^{n-1+2k}} \\
&= -x^{-n} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(k-1)!(k-1+n+1)!} \frac{x^{n+1+2(k-1)}}{2^{n+1+2(k-1)}} \\
&= -x^{-n} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+n+1)!} \frac{x^{n+1+2k}}{2^{n+1+2k}} \\
&= -x^{-n} J_{n+1}(x).
\end{aligned}$$

From formula (1) we get

$$x^n J_n'(x) + n x^{n-1} J_n(x) = x^n J_{n-1}(x)$$

and dividing both sides by x^n gives

$$J_n'(x) + \frac{n}{x} J_n(x) = J_{n-1}(x)$$

which is (3).

From formula (2) we get

$$x^{-n}J_n'(x) - nx^{-(n+1)}J_n(x) = -x^{-n}J_{n+1}(x)$$

and multiplying both sides by x^n we get

$$J_n'(x) - \frac{n}{x}J_n(x) = -J_{n+1}(x)$$

which is (4).
Add (3) and (4) together and get (5):

$$2J_n'(x) + J(n-1)(x) - J_{n+1}(x)$$

because the $(n/x)J_n(x)$ terms cancel.
Likewise, subtract (4) from (3) to get (6):

$$\frac{2n}{x}J_n(x) = J_{n-1}(x) + J_{n+1}(x).$$

2. Multiply the Bessel equation

$$u'' + \frac{1}{x}u' + \left(\alpha^2 - \frac{n^2}{x^2}\right)u = 0$$

by $2x^2u'$ and integrate from 0 to 1 and show that

$$\int_0^1 xJ_n(\alpha x)^2\,dx = \frac{1}{2}J_n'(\alpha)^2 + \frac{1}{2}\left(1 - \frac{n^2}{\alpha^2}\right)J_n(\alpha)^2.$$

Then put $\alpha = z_{nm}$ and conclude (using formula (4)) that

$$\int_0^1 xJ_n(z_{nm}x)^2\,dx = \frac{1}{2}J_n'(z_{nm})^2 = \frac{1}{2}J_{n+1}(z_{nm})^2.$$

Recall that $u(x) = J_n(\alpha x)$ is a solution of the differential equation. Multiply the equation by $2x^2u'$ and integrate from 0 to 1 and get:

$$\begin{aligned} 0 &= \int_0^1 2x^2u'u'' + 2x(u')^2 + 2(\alpha^2x^2 - n^2)uu'\,dx \\ &= \int_0^1 (x^2(u')^2)' + (\alpha^2x^2 - n^2)(u^2)'\,dx \\ &= \int_0^1 (x^2(u')^2)' + \alpha(x^2u^2)' - 2\alpha xu^2 - n^2(u^2)'\,dx \\ &= [x^2(u')^2 + \alpha x^2u^2 - n^2u^2]\Big|_0^1 - 2\alpha\int_0^1 xu^2\,dx \end{aligned}$$

where the clever step comes after the third equals sign.

Now we recall that $u(x) = J_n(\alpha x)$, so that $u'(x) = \alpha J_n'(\alpha x)$, and move the integral over to the other side of the equation and get

$$\int_0^1 x J_n(\alpha x)^2\,dx = \frac{1}{2}J_n'(\alpha)^2 + \frac{1}{2}\left(1 - \frac{n^2}{\alpha}\right) J_n(\alpha)^2.$$

But if $\alpha = z_{nm}$ is a place where J_n is zero, then we have

$$\int_0^1 x J_n(z_{nm}x)^2\,dx = \frac{1}{2}J_n'(z_{nm})^2 = \frac{1}{2}J_{n+1}(z_{nm})^2,$$

using formula (4) (and the fact that $J_n(z_{nm}) = 0$ again).

3. Calculate the coefficients c_{nm} and d_{nm} in the solution of the wave equation.

We start from the formula for u at the top of page 7 of the notes:

$$\begin{aligned} u(r,\theta,t) &= \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} J_n(\sqrt{\lambda_{nm}}r)\cos(\sqrt{\lambda_{nm}}ct)\left(a_{nm}\cos n\theta + b_{nm}\sin n\theta\right) \\ &\quad + \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} J_n(\sqrt{\lambda_{nm}}r)\sin(\sqrt{\lambda_{nm}}ct)\left(c_{nm}\cos n\theta + d_{nm}\sin n\theta\right) \end{aligned}$$

To calculate the coefficients c_{nm} and d_{nm}, we need to take the derivative of u with respect to t and set $t = 0$ (which gives us the initial condition containing g):

$$g(r,\theta) = u_t(r,\theta,0) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} \sqrt{\lambda_{nm}}cJ_n(\sqrt{\lambda_{nm}}r)(c_{nm}\cos n\theta + d_{nm}\sin n\theta)$$

If we view θ as the variable and r as constant for the moment, this becomes an ordinary Fourier series for $g(r,\theta)$, so we have

$$\begin{aligned} \sum_{m=1}^{\infty} \sqrt{\lambda_{0m}}\,c\,c_{0m}J_0(\sqrt{\lambda_{0m}}r) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} g(r,\theta)\,d\theta \quad \text{for } n = 0, \\ \sum_{m=1}^{\infty} \sqrt{\lambda_{nm}}\,c\,c_{nm}J_n(\sqrt{\lambda_{nm}}r) &= \frac{1}{\pi}\int_{-\pi}^{\pi} g(r,\theta)\cos m\theta\,d\theta \quad \text{for } n \geq 1, \\ \sum_{m=1}^{\infty} \sqrt{\lambda_{nm}}\,c\,d_{nm}J_n(\sqrt{\lambda_{nm}}r) &= \frac{1}{\pi}\int_{-\pi}^{\pi} g(r,\theta)\sin m\theta\,d\theta \quad \text{for } n \geq 1. \end{aligned}$$

But the left sides of these are Fourier-Bessel series, so using the results of the "Orthogonality" section of the notes we finally obtain the coefficients:

$$c_{0m} = \frac{\dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi}\int_0^a rg(r,\theta)J_0(\sqrt{\lambda_{0m}}r)\,dr\,d\theta}{\sqrt{\lambda_{nm}}\,c\displaystyle\int_0^a rJ_0(\sqrt{\lambda_{0m}}\,r)^2\,dr} \qquad \text{for } n=0, m\geq 1$$

$$c_{nm} = \frac{\dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi}\int_0^a rg(r,\theta)J_n(\sqrt{\lambda_{nm}}r)\cos n\theta\,dr\,d\theta}{\sqrt{\lambda_{nm}}\,c\displaystyle\int_0^a rJ_n(\sqrt{\lambda_{nm}}\,r)^2\,dr} \qquad \text{for } n\geq 1, m\geq 1$$

$$d_{nm} = \frac{\dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi}\int_0^a rg(r,\theta)J_n(\sqrt{\lambda_{nm}}r)\sin n\theta\,dr\,d\theta}{\sqrt{\lambda_{nm}}\,c\displaystyle\int_0^a rJ_n(\sqrt{\lambda_{nm}}\,r)^2\,dr} \qquad \text{for } n\geq 1, m\geq 1$$

and the denominators are given by

$$\int_0^a rJ_n(\sqrt{\lambda_{nm}}\,r)^2\,dr = \int_0^a rJ_n\left(\frac{z_{nm}}{a}r\right)^2\,dr = \frac{a^2}{2}J_{n+1}(z_{nm})^2.$$

4. Prove the formula

$$\int_0^a x^{n+1}J_n\left(\frac{\alpha x}{a}\right)\,dx = \frac{a^{n+2}}{\alpha}J_{n+1}(\alpha)$$

and use it to solve the problem (with circular symmetry, so there's no dependence on θ):

$$\frac{\partial^2 u}{\partial t^2} = 16\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right), \quad 0<r<1,\ t>0$$

with boundary condition $u(1,t)=0$ and initial conditions $u(r,0)=1-r^2$ and $u_t(r,0)=1$.

(*Hint*: I think the answer is

$$u(r,t) = \sum_{m=1}^{\infty} J_0(z_m r)\left[\frac{8}{z_m^3 J_1(z_m)}\cos(4z_m t) + \frac{1}{2z_m^2 J_1(z_m)}\sin(4z_m t)\right].$$

where z_m is the mth positive zero of the Bessel function $J_0(x)$. You'll need to use identities (1) and (6) and integration by parts to get it into this form.)

Formula (1) from problem 1 is

$$\frac{d}{dx}(x^n J_n(x)) = x^n J_{n-1}(x) \quad \text{for} \quad n \geq 1$$

Replace n with $n+1$ and get

$$\frac{d}{dx}(x^{n+1} J_{n+1}(x)) = x^{n+1} J_n(x) \quad \text{for} \quad n \geq 0$$

and turn it around into the integral formula

$$\int_0^a x^{n+1} J_n(x)\, dx = \left[x^{n+1} J_{n+1}(x)\right]\Big|_0^a = a^{n+1} J_{n+1}(a).$$

Now to do the integral

$$\int_0^a x^{n+1} J_n\left(\frac{\alpha x}{a}\right) dx$$

we let $u = \alpha x/a$, so $dx = (a/\alpha)\, du$ and we get

$$\begin{aligned}
\int_0^a x^{n+1} J_n\left(\frac{\alpha x}{a}\right) dx &= \frac{a}{\alpha}\int_0^\alpha \left(\frac{a}{\alpha}u\right)^{n+1} J_n(u)\, du \\
&= \left(\frac{a}{\alpha}\right)^{n+2} \int_0^\alpha u^{n+1} J_n(u)\, du \\
&= \left(\frac{a}{\alpha}\right)^{n+2} \alpha^{n+1} J_{n+1}(\alpha) = \frac{a^{n+2}}{\alpha} J_{n+1}(\alpha)
\end{aligned}$$

which is the formula we needed to prove.

Now we're ready to solve the PDE. You could either start from scratch and separate variables (we're looking for $u(r,t)$ since there's no dependence on θ, or else we can use the solution from the "back to the wave equation" section of the notes, and just get rid of all the terms that have θ in them, namely all the terms with $n \geq 1$. So the solution is

$$u(r,t) = \sum_{m=1}^{\infty} a_{0m} J_0(\sqrt{\lambda_{0m}}r)\cos(4\sqrt{\lambda_{0m}}t) + \sum_{m=1}^{\infty} c_{0m} J_0(\sqrt{\lambda_{0m}}r)\sin(4\sqrt{\lambda_{0m}}t)$$

since $c = 4$, $a = 1$ and $\lambda_{0m} = z_{0m}^2$ (where z_{0m} is the mth positive zero of the Bessel function $J_0(x)$). Since we're only using J_0 in this problem, we'll just write λ_m instead of λ_{0m}, z_m instead of z_{0m}, and a_m and c_m instead of a_{0m} and c_{0m}.

Since the initial conditions are $f(r) = 1 - r^2$ and $g(r) = 1$, we also know that

$$a_m = \frac{\displaystyle\int_0^1 r(1-r^2) J_0(z_m r)\, dr}{\displaystyle\int_0^1 r J_0(z_m r)^2\, dr} \quad \text{for } m \geq 1$$

and (from problem 3),

$$c_m = \frac{\displaystyle\int_0^1 r J_0(z_m r)\, dr}{\displaystyle 4z_m \int_0^1 r J_0(z_m r)^2\, dr} \quad \text{for } m \geq 1$$

and the integrals in the denominators evaluate to $\frac{1}{2}J_1(z_m)^2$ (from problem 2).

To evaluate the integral in the numerator of a_m, we first make the substitution $x = z_m r$ (so $dr = dx/z_m$ and rewrite it as

$$\int_0^{z_m} \frac{x}{z_m}\left(1-\frac{x^2}{z_m^2}\right) J_0(x)\frac{dx}{z_m} = \frac{1}{z_m^4}\int_0^{z_m} x(z_m^2-x^2)J_0(x)\,dx.$$

Next, integrate by parts with $u = z_m^2 - x^2$ and $dv = xJ_0(x)\,dx$. Then from formula (1) in the notes (with $n = 1$), $v = xJ_1(x)$ so we get

$$\begin{aligned}\int_0^1 r(1-r^2)J_0(z_m r)\,dr &= \frac{1}{z_m^4}\int_0^{z_m} x(z_m^2-x^2)J_0(x)\,dx \\ &= \frac{1}{z_m^4}[(z_m^2-x^2)xJ_1(x)]\Big|_0^{z_m} + \frac{1}{z_m^4}\int_0^{z_m} 2x^2J_1(x)\,dx \\ &= \frac{2}{z_m^4}\int_0^{z_m} x^2J_1(x)\,dx \\ &= \frac{2}{z_m^4}\,x^2J_2(x)\Big|_0^{z_m} = \frac{2}{z_m^2}J_2(z_m)\end{aligned}$$

where we used (1) again, this time with $n = 2$. Therefore, we have that

$$a_m = \frac{\int_0^1 r(1-r^2)J_0(z_m r)\,dr}{\int_0^1 rJ_0(z_m r)^2\,dr} = \frac{\frac{2}{z_m^2}J_2(z_m)}{\frac{1}{2}J_1(z_m)^2} = \frac{4J_2(z_m)}{z_m^2J_1(z_m)^2}.$$

We can do a little better still — from equation (6) with $n = 1$, we can replace $J_2(z_m)$ with $2J_1(z_m)/z_m$, and write:

$$a_m = \frac{8}{z_m^3J_1(z_m)},$$

and that's about as simple as we can get it.
For c_m, we need only use our substitution $x = z_m r$ and formula (1) to integrate

$$\int_0^1 rJ_0(z_m r)\,dr = \int_0^{z_m}\frac{x}{z_m}J_0(x)\frac{dx}{z_m} = \frac{1}{z_m^2}z_mJ_1(z_m) = \frac{J_1(z_m)}{z_m}$$

Therefore

$$c_m = \frac{\int_0^1 rJ_0(z_m r)\,dr}{4z_m\int_0^1 rJ_0(z_m r)^2\,dr} = \frac{\frac{J_1(z_m)}{z_m}}{4z_m\frac{1}{2}J_1(z_m)^2} = \frac{1}{2z_m^2J_1(z_m)}$$

Therefore, the solution is

$$u(r,t) = \sum_{m=1}^{\infty} J_0(z_m r)\left[\frac{8}{z_m^3J_1(z_m)}\cos(4z_m t) + \frac{1}{2z_m^2J_1(z_m)}\sin(4z_m t)\right].$$

Whew!