Math 425 / AMCS 525 **Dr. DeTurck**
Practice problems for midterm 2 **March 23, 2016**

1. Say that a function is *oddly odd* if it satisfies both the conditions

$$f(-x) = -f(x), \qquad f(L+x) = f(L-x)$$

(a) Show that such a function is periodic with period $4L$.

(b) Draw the graph of a non-zero oddly odd function for $-5L \le x \le 5L$ (pick one that is interesting but not too interesting, perhaps have the graph consist mostly of line segments). What (if any) kind of symmetry does it have around the line $x = L$? ...around the line $x = 2L$?

(c) Show that the Fourier series of an oddly odd function is of the form

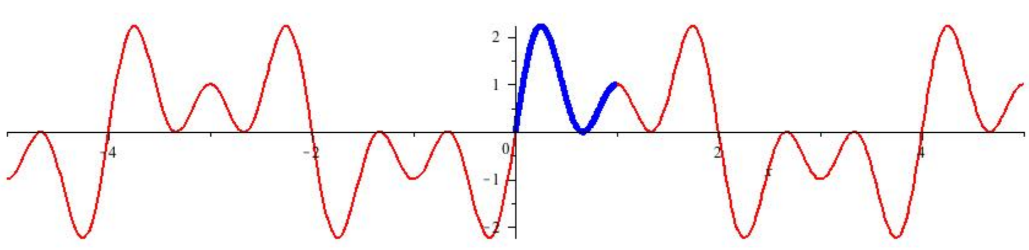
$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{(2n-1)\pi x}{2L}.$$

Give a formula for the coefficients b_n.

(a) Repeatedly use the given properties. We have to show that $f(x+4L) = f(x)$ for all x. Well,

$$\begin{aligned} f(x+4L) &= f(L+(x+3L)) && \text{getting ready to use the second condition} \\ &= f(L-(x+3L)) && \text{by the second condition} \\ &= f(-(x+2L)) && \\ &= -f(x+2L) && \text{by the first condition} \\ &= -f(L+(x+L)) && \text{getting ready to use the second condition} \\ &= -f(L-(x+L) && \text{by the second condition} \\ &= -f(-x) && \\ &= f(x) && \text{by the first condition} \end{aligned}$$

(b)



In this figure, $L = 1$. The function is even around $x = L$ (i.e., $f(L+x) = f(L-x)$) and odd around $x = 2L$ (i.e., $f(2L+x) = -f(2L-x)$).

(c) We have $b_n = 0$ if n is even, and if n is odd, say $n = 2k - 1$,

$$b_{2k-1} = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{(2k-1)\pi x}{2L}\right)\,dx$$

and so

$$f(x) = \sum_{k=1}^{\infty} b_{2k-1}\sin\left(\frac{(2k-1)\pi x}{2L}\right)$$

with these coefficients.

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2. Let

$$f(x) = \begin{cases} x & 0 < x < 1 \\ 2 - x & 1 < x < 2 \end{cases}$$

Solve the heat equation $u_t = u_{xx}$ for $x \in [0, 2]$ and $t \in [0, \infty)$ with initial condition $u(x, 0) = f(x)$ and boundary conditions $u(0, t) = u(2, t) = 0$. Draw a sketch of the graph of $u(x, \epsilon)$ for a fixed, very small value of ϵ and $0 \le x \le 2$.

Because of the boundary conditions, we know we have to use sines in our Fourier expansion. So the solution will be

$$u(x, t) = \sum_{n=1}^{\infty} b_n e^{-n^2\pi^2 t/4}\sin\frac{n\pi x}{2},$$

where

$$b_n = \frac{\langle f(x), \sin\frac{n\pi x}{2}\rangle}{\langle \sin\frac{n\pi x}{2}, \sin\frac{n\pi x}{2}\rangle}$$

and the inner product is given by

$$\langle f, g\rangle = \int_0^2 f(x)\sin\frac{n\pi x}{2}\,dx.$$

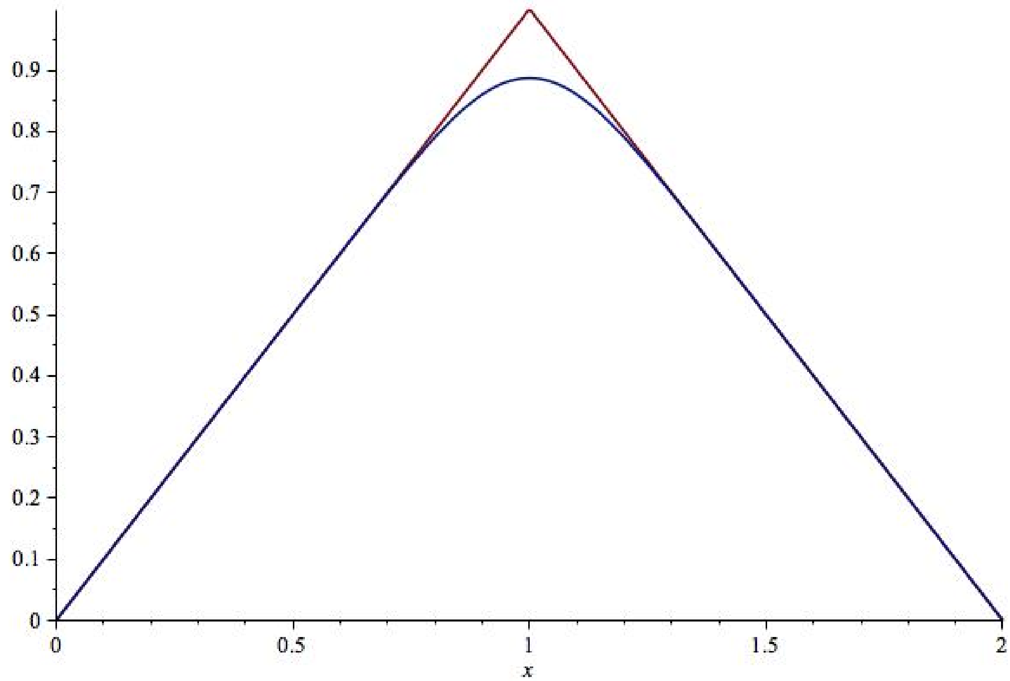
Since f is even around 1, you can see that $b_n = 0$ if n is even. For odd n, the integral from 0 to 2 will be twice what it is from 0 to 1. So we get

$$b_{2k+1} = \frac{(-1)^k 8}{(2k+1)^2\pi^2},$$

and so

$$u(x, t) = \sum_{k=0}^{\infty} \frac{(-1)^k 8}{(2k+1)^2\pi^2} e^{-(2k+1)^2\pi^2 t/4}\sin\frac{(2k+1)\pi x}{2}.$$

In this figure:



the red curve is the initial data and the blue curve is the solution at time $t = 0.01$ – you can see that the corner gets around off and the whole curve goes down a little bit.

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3. (a) Find the Fourier (cosine) series of the function $f(x) = x^2$, $-\pi < x < \pi$.

(b) Draw the graph of the function to which your series converges. Explain how you know the series converges pointwise to this function. Does it converge uniformly?

(c) Use the series to show that

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} + \cdots = \frac{\pi^2}{6}$$

and

$$1 - \frac{1}{2^2} + \frac{1}{3^2} - \cdots + \frac{(-1)^{n+1}}{n^2} + \cdots = \frac{\pi^2}{12}$$

(d) Use the results in part (c) to deduce

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots + \frac{1}{(2n-1)^2} + \cdots = \frac{\pi^2}{8}$$

(a) Since x^2 is even, and extends to be continuous as a periodic function with period 2π, we'll have

$$x^2 = \sum_{n=0}^{\infty} a_n \cos nx$$

for $-\pi < x < \pi$ where

$$a_n = \frac{\langle x^2, \cos nx \rangle}{\langle \cos nx, \cos nx \rangle}$$

and

$$\langle f, g \rangle = \int_0^{\pi} f(x)g(x)\, dx.$$

(Using an integral from 0 to 2π will double both the numerator and the denominator in a_n, so won't affect its value). Thus

$$a_n = \frac{2}{\pi} \int_0^{\pi} x^2 \cos nx\, dx = \frac{(-1)^n 4}{n^2}$$

for $n > 0$ and

$$a_0 = \frac{1}{\pi} \int_0^{\pi} x^2\, dx = \frac{\pi^2}{3}.$$

So

$$x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{(-1)^n 4}{n^2} \cos nx.$$

(b) The series does converge uniformly (by the theorem since the function is continous and piecewise differentiable, or by the Weierstrass M-test.
(c) Plug in $x = \pi$ and $x = 0$ to get the results in this part. Add the two together and divide by 2 to get the result in (d).

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4. Solve the initial-boundary value problem for the wave equation:

$$u_{tt} = c^2 u_{xx}, \quad 0 < x < 1, \quad t > 0$$

where $u(x, 0) = \sin \pi x$, $u_t(x, 0) = 0$, $u(0, t) = 0$, $u(1, t) = 1$.

The wrinkle in this problem is the inhomogeneous boundary conditions $u(1, t) = 1$. So we begin by writing $u(x, t) = v(x, t) + w(x, t)$, where v takes care of the inhomogeneous boundary condition, and u picks up the slack. The simplest function that satisfies the wave equation together with $v(0, t) = 0$ and $v(1, t) = 1$ is $v(x, t) = x$. So we let $w(x, t) = u(x, t) - x$. Then w will solve the problem:

$$w_{tt} = c^2 w_{xx}$$

together with the initial condition $w(x, 0) = \sin \pi x - x$ and the boundary conditions $w(0, t) = w(1, t) = 0$. By the usual separation of variables shtick, we have

$$w(x, t) = \sum_{n=1}^{\infty} b_n \cos(cn\pi t) \sin(n\pi x),$$

where

$$b_n = \frac{\langle \sin \pi x - x, \sin n\pi x \rangle}{\langle \sin n\pi x, \sin n\pi x \rangle}$$

and

$$\langle f, g \rangle = \int_0^1 f(x)g(x)\,dx.$$

Thus

$$b_n = 2\int_0^1 (\sin \pi x - x)\sin n\pi x\,dx = \begin{cases} 1 + \dfrac{(-1)^n 2}{n\pi} & \text{if } n = 1 \\ \dfrac{(-1)^n 2}{n\pi} & \text{if } n > 1 \end{cases}$$

Putting it all together, we get that

$$u(x,t) = x + \cos(c\pi t)\sin(\pi x) + \sum_{n=1}^{\infty} \frac{(-1)^n 2}{n\pi}\cos(cn\pi t)\sin(n\pi x).$$

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5. (a) Find the eigenvalues and eigenfunctions of the boundary-value problem:

$$u'' + \lambda u = 0, \qquad u(0) = 0, \qquad u'(3) + u(3) = 0$$

for $u(x)$ defined on the interval [0,3].

(b) If we number the eigenvalues in increasing order, so that $\lambda_1 < \lambda_2 < \lambda_3 < ...$, find A and B so that

$$\lim_{n\to\infty} \left(\lambda_n - (An + B)^2\right) = 0.$$

(a) First check to see if λ could be negative: If $\lambda < 0$, because $u(0) = 0$, we'd have $u(x) = c\sinh\sqrt{-\lambda}\,x$. But then

$$u'(3) + u(3) = c\Big(\sqrt{-\lambda}\cosh(3\sqrt{-\lambda}) + \sinh(3\sqrt{-\lambda})\Big)$$

and this can never be zero, since $\cosh x > 0$ and $\sinh x > 0$ if $x > 0$.
Likewise, λ can't be zero, since no linear function other than $u = 0$ satisfies the conditions.
Thus λ is positive and $u(x) = c\sin\sqrt{\lambda}\,x$ (because $u(0) = 0$). To find the precise eigenvalues, we need to solve for λ:

$$u'(3) + u(3) = \sqrt{\lambda}\cos 3\sqrt{\lambda} + \sin 2\sqrt{\lambda} = 0.$$

Rewrite this as

$$\tan 3\sqrt{\lambda} = -\sqrt{\lambda}.$$

Graphing both sides, with $\sqrt{\lambda}$ on the horizontal axis and the values of the two sides on the vertical, we see that there are infinitely many solutions, places where the line $y = -\sqrt{\lambda}$ crosses the graph of $y = \tan 3\sqrt{\lambda}$, one for each branch of the tangent function. This gives us the eigenvalues, and the corresponding eigenfunctions are $\sin\sqrt{\lambda}\,x$.
(b) As $\sqrt{\lambda}$ gets larger and larger, the places where the graph of $y = -\sqrt{\lambda}$ crosses the graph of $y = \tan 3\sqrt{\lambda}$ get closer and closer to the vertical asymptotes of the tangent function, which occur for

$$3\sqrt{\lambda} = \frac{(2n+1)\pi}{2}.$$

In other words, the larger n is, the closer λ_n is to

$$\left(\frac{(2n+1)\pi}{6}\right)^2.$$

So $A = \frac{\pi}{3}$ and $B = \frac{\pi}{6}$.

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6. This problem shouldn't require any integration, but (b) and especially (c) will require some thinking.

(a) Solve the Laplace equation

$$u_{rr} + \frac{u_r}{r} + \frac{u_{\theta\theta}}{r^2} = 0$$

on the inside of the disk $r < 2$ with boundary condition

$$u(2, \theta) = 8 \sin(3\theta)$$

for $0 < \theta < 2\pi$.

(b) Solve the Laplace equation on the *outside* of the circle $r = 1$ (that is, for $r > 1$) with boundary condition

$$u(1, \theta) = \sin(2\theta).$$

Assume we want the solution to remain bounded as $r \to \infty$. How does this change the form of the solution?

(c) Solve the Laplace equation in the *annulus* inside the circle $r = 2$ but outside the circle $r = 1$, i.e., for $1 < r < 2$ with boundary conditions

$$u(1, \theta) = \sin(2\theta) \qquad u(2, \theta) = 8 \sin(3\theta).$$

Since there is neither a condition at $r = 0$ nor at infinity, both parts of $R(r)$ in the separated solutions come into play.

(a) On the *inside* of the disk, the solution is

$$u(r, \theta) = a_0 + \sum_{n=1}^{\infty} \left(\frac{r}{2}\right)^n (a_n \cos n\theta + b_n \sin n\theta).$$

To match the boundary condition when $r = 2$, we all the a_n's to be zero as well as all the b_n's except for $n = 3$, and $b_3 = 8$. So

$$u(r, \theta) = 8 \left(\frac{r}{2}\right)^3 \sin 3\theta = r^3 \sin 3\theta.$$

(b) On the *outside* of the disk, the solution is

$$u(r, \theta) = a_0 + \sum_{n=1}^{\infty} \left(\frac{1}{r}\right)^n (a_n \cos n\theta + b_n \sin n\theta).$$

This time, to match the boundary conditions we just need $b_2 = 1$ and the rest zero. So

$$u(r, \theta) = \frac{\sin 2\theta}{r^2}.$$

(c) For this one, you might think we just need to add the previous two solutions, but that wouldn't be right, since neither solution is zero on the other's boundary. But now, we have both the interior and exterior parts of the solution, so u is of the form:

$$u(r, \theta) = a_0 + \sum_{n=1}^{\infty} \left(\frac{r}{2}\right)^n (a_n \cos n\theta + b_n \sin n\theta) + \left(\frac{1}{r}\right)^n (c_n \cos n\theta + d_n \sin n\theta).$$

Given all the data, the only non-zero coefficients will be b_2, b_3, d_2 and d_3. So we have

$$u(r, \theta) = b_2 \left(\frac{r}{2}\right)^2 \sin 2\theta + b_3 \left(\frac{r}{2}\right)^3 \sin 3\theta + d_2 \frac{\sin 2\theta}{r^2} + d_3 \frac{\sin 3\theta}{r^3}$$

When $r = 1$, the boundary data says:

$$\sin 2\theta = \frac{b_2}{4} \sin 2\theta + \frac{b_3}{8} \sin 3\theta + d_2 \sin 2\theta + d_3 \sin 3\theta$$

which tells us that $\frac{1}{4} b_2 + d_2 = 1$ and $\frac{1}{8} b_3 + d_3 = 0$. Likewise, when $r = 2$, we have

$$8 \sin 3\theta = b_2 \sin 2\theta + b_3 \sin 3\theta + \frac{d_2}{4} \sin 2\theta + \frac{d_3}{8} \sin 3\theta$$

which tells us that $b_2 + \frac{1}{4} d_2 = 0$ and $b_3 + \frac{1}{8} d_3 = 8$. So there is a 2-by-2 system for b_2 and d_2, and another for b_3 and d_3. We have

$$\begin{bmatrix} b_2 \\ d_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & 1 \\ 1 & \frac{1}{4} \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = -\frac{16}{15} \begin{bmatrix} \frac{1}{4} & -1 \\ -1 & \frac{1}{4} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{4}{15} \\ \frac{16}{15} \end{bmatrix}$$

and

$$\begin{bmatrix} b_3 \\ d_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{8} & 1 \\ 1 & \frac{1}{8} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 8 \end{bmatrix} = -\frac{64}{63} \begin{bmatrix} \frac{1}{8} & -1 \\ -1 & \frac{1}{8} \end{bmatrix} \begin{bmatrix} 0 \\ 8 \end{bmatrix} = \begin{bmatrix} \frac{512}{63} \\ -\frac{64}{63} \end{bmatrix}$$

Therefore

$$u(r, \theta) = -\frac{4}{15} \left(\frac{r}{2}\right)^2 \sin 2\theta + \frac{512}{63} \left(\frac{r}{2}\right)^3 \sin 3\theta + \frac{16}{15} \frac{\sin 2\theta}{r^2} - \frac{64}{63} \frac{\sin 3\theta}{r^3}$$

(sorry about the numbers!)

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7. For the exam, be sure you know the integrals:

$$\int \cos ax \, \cos bx \, dx, \qquad \int \sin ax \, \cos bx \, dx.$$